UNIVERSITY OF CAPE TOWN
IYUNIVESITHI YASEKAPA • UNIVERSITEIT VAN KAAPSTAD

International Conferences on

Computability & Complexity in Analysis (CCA 2011)
Computability, Complexity & Randomness (CCR 2011)

University of Cape Town, 31 January-4 February 2011

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Preface

The eighth International Conference on Computability and Complexity in Analysis (CCA 2011) and the sixth International Conference on Computability, Complexity and Randomness (CCR 2011) take place jointly at the University of Cape Town (UCT) in South Africa from 31 January–4 February 2011.

Both events are continuations of successful conference series that helped to gather researchers in the respective fields and to catalyse activities and collaborations. Whereas the conference CCA 2011 deals with computations on real numbers from many different perspectives, such as those of computability, complexity, constructivity and randomness, the conference CCR 2011 focuses on algorithmic randomness from many different directions such as computability, complexity, logic and applications in computer science.

We expect that this joint event will reveal close links between the domains of both conferences and, certainly, it will inspire new interactions. Already the submitted papers indicate that the boundaries between both topics are open and several papers fit equally well into both conferences. In fact, we have decided to run the joint conference in one session with presentations not separated according to conferences, but grouped slightly according to topics.

The number of participants of the joint conferences reflect the highly international research activities in this field, they come from: South Africa (12), Germany (10), USA (8), Japan (5), New Zealand (5), France (4), UK (3), Russia (2), Argentina (1), Australia (1), Canada (1), Czech Republic (1) and Iran (1).[1] For South African researchers and students this conference is a unique opportunity to get in contact with leading international researchers in the respective fields.

On behalf of the organisers we would like to acknowledge the generous support of these conferences, provided by the University of Cape Town, the sponsors Dimension Data and Internet Solutions and the student support offered by the Association for Symbolic Logic. Without this support, this double conference would not have been possible.

Finally, we would like to thank the invited speakers, the authors and co-authors and the Programme Committee for their enthusiastic support of this event!

Peter Hertling *(for the CCA 2011 Programme Committee)*
Rod Downey *(for the CCR 2011 Programme Committee)*
Vasco Brattka *(for the joint Organising Committee)*

[1] Numbers as of 20 January 2010.

Computability and Complexity in Analysis (CCA 2011)

Invited Speakers

- Abbas Edalat, London, UK
- Mathieu Hoyrup, Nancy, France
- Matthias Schröder, Munich, Germany
- Peter Schuster, Leeds, UK
- Stephen G. Simpson, Pennsylvania, USA
- Michael Yampolsky, Toronto, Canada

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- Alex Simpson, Edinburgh, UK
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- Vasco Brattka, Cape Town, South Africa (chair)
- Holger Spakowski, Cape Town, South Africa

Assistants

- Thomas Birch, Cape Town, South Africa
- Adrian Frith, Cape Town, South Africa

Computability, Complexity and Randomness (CCR 2011)

Invited Speakers

- Verónica Becher, Buenos Aires, Argentina
- Willem Fouché, Pretoria, South Africa
- Noam Greenberg, Wellington, New Zealand
- Wolfgang Merkle, Heidelberg, Germany
- André Nies, Auckland, New Zealand
- Theodore A. Slaman, Berkeley, USA

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- Rod Downey, Wellington, New Zealand (co-chair)
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- Bjørn Kjos-Hanssen, Hawai'i, USA
- Antonín Kučera, Prague, Czech Republic
- Steffen Lempp, Madison, USA
- Jan Reimann, Pennsylvania, USA

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Assistants

- Thomas Birch, Cape Town, South Africa
- Adrian Frith, Cape Town, South Africa

Talks

A better complexity of finite sequences

Verónica Becher (Invited Speaker)
Universidad de Buenos Aires, Argentina
(joint work with Pablo Ariel Heiber)

We present a new complexity function defined from combinatorial properties of strings, and we prove that not only it satisfies fundamental properties of program-size complexity, but also it is computable in linear time and space.

Our complexity function is defined by giving a score for each symbol in a string according to how many new substrings are given raised by that symbol. The accumulation of the single scores yields a global indicator of the repeated substrings in the given string. The most complex strings are those having the largest number of different substrings, the de Bruijn strings. The least complex are the runs of a single symbol.

Our function is monotone in the prefix ordering of strings, and subadditive for concatenation. We prove an upper bound of the number of strings with complexity up to a given value, and we show that most strings of any given length have almost maximal complexity.

With these results we argue for the usefulness our complexity function in practice, as it overcomes the drawbacks of Lempel Ziv complexity, the lack of subadditivity of approximations of Kolmogorov complexity based on compressors, and the absence of efficient algorithms to compute resource-bounded versions of Kolmogorov complexity.

The Bolzano-Weierstraß Theorem is the Jump of Weak Kőnig's Lemma

Vasco Brattka
University of Cape Town, South Africa
(joint work with Guido Gherardi and Alberto Marcone)

We classify the computational content of the Bolzano-Weierstraß Theorem and variants thereof in the Weihrauch lattice. For this purpose we first introduce the concept of a derivative or jump in this lattice and we show that it has some properties similar to the Turing jump. Using this concept we prove that the derivative of closed choice of a computable metric space is the cluster point problem of that space. By specialization to sequences with a relatively compact range we obtain a characterization of the Bolzano-Weierstraß Theorem as the derivative of compact choice. In particular, this shows that the Bolzano-Weierstraß Theorem on real numbers is the jump of Weak Kőnig's Lemma. Likewise, the Bolzano-Weierstraß Theorem on the binary space is the jump of the lesser limited principle of omniscience LLPO and the Bolzano-Weierstraß Theorem on natural numbers can be characterized as the jump of the idempotent closure of LLPO. We also introduce the compositional product of two Weihrauch degrees f and g as the supremum of the composition of any two functions below f and g, respectively. Using this concept we can express the main result such that the Bolzano-Weierstraß Theorem is the compositional product of Weak Kőnig's Lemma and the Monotone Convergence Theorem. We also study the class of weakly limit computable functions, which are functions that can be obtained by composition of weakly computable functions with limit computable functions. We prove that the Bolzano-Weierstraß Theorem on real numbers is complete for this class. Likewise, the unique cluster point problem on real numbers is complete for the class of functions that are limit computable with finitely many mind changes. We also prove that the Bolzano-Weierstraß Theorem on real numbers and, more generally, the unbounded cluster point problem on real numbers is uniformly low limit computable. Finally, we also provide some separation techniques that allow to prove non-reducibilities between certain variants of the Bolzano-Weierstraß Theorem.

Weak-operator Continuous Linear Functionals

Douglas S. Bridges
University of Canterbury, New Zealand

Let H be a Hilbert space, and $\mathcal{B}(H)$ the algebra of bounded operators on H. Working entirely within a (Bishop-style) constructive framework, we prove two characterisations of weak-operator continuous linear functionals on subspaces of $\mathcal{B}(H)$.

Theorem 1 *Let $\mathcal{R}$ be a linear subspace of $\mathcal{B}(H)$ with weak-operator totally bounded unit ball, and let u be a linear functional on $\mathcal{R}$. Suppose that*

(i) the unit ball of $\mathcal{R}$ is weak-operator totally bounded,

(ii) there exist $\xi, \zeta \in H_\infty$ and $\delta > 0$ such that $|u(T)| \leqslant \delta \left|\sum_{n=1}^{\infty} \langle T\xi_n, \zeta_n\rangle\right|$ for all $T \in \mathcal{R}$, and

(iii) $\sup\left\{\left|\sum_{n=1}^{\infty} \langle T\xi_n, \zeta_n\rangle\right| : T \in \mathcal{R}_1\right\} > 0$.

Then there exists $\lambda \in \mathbf{C}$ such that $u(T) = \lambda \sum_{n=1}^{\infty} \langle T\xi_n, \zeta_n\rangle$ for all $T \in \mathcal{R}$.

Theorem 2 *Let H be a nontrivial Hilbert space, and u a weak-operator continuous linear functional on $\mathcal{B}(H)$. Let $\xi_1, \ldots, \xi_N$ and $\zeta_1, \cdots, \zeta_N$ be nonzero vectors in H, and δ a positive number, such that $|u(T)| \leqslant \delta \sum_{n=1}^{N} |\langle T\xi_n, \zeta_n\rangle|$ for all $T \in \mathcal{B}(H)$. Then there exists $\mathbf{x} \in \mathbf{C}\xi_1 \times \cdots \times \mathbf{C}\xi_N$ such that $u(T) = \sum_{n=1}^{N} \langle Tx_n, \zeta_n\rangle$ for all $T \in \mathcal{B}(H)$.*

This theorem requires a sequence of lemmas, some with long, complicated, and highly functional-analytic proofs.

Transcendence of irrational automatic numbers

Yann Bugeaud
Université de Strasbourg, France
(joint work with Boris Adamczewski)

In 1965, Hartmanis and Stearns proposed to define the complexity of real numbers, by emphasizing the quantitative aspect of the notion of calculability introduced by Turing. They posed the following problem: Do there exist irrational algebraic numbers which are computable in real time? In 1968, Cobham suggested to restrict this problem to a particular class of Turing machines, namely to finite automata. After several attempts by Cobham in 1968 and by Loxton and van der Poorten in 1982, Loxton and van der Poorten finally claimed to have completely solved the restricted problem in 1988, but the proof they proposed contains a gap.

In a joint work with Adamczewski [On the complexity of algebraic numbers I. Expansions in integer bases. Annals of Math. 165 (2007), 547-565], we have established Cobham's conjecture.

Theorem 3 *Let $b \geq 2$ be an integer. The b-ary expansion of an irrational algebraic number cannot be generated by a finite automaton. In other words, irrational automatic numbers are transcendental.*

In the same work, we give a lower bound for the subword complexity of the b-ary expansion of an irrational algebraic number.

Layerwise computable properties of complex oscillations

George Davie
Unisa, South Africa
(joint work with W.L. Fouché)

In, The descriptive complexity of Brownian motion Adv. Math., 155 (2000), Fouché showed how to build a generic Brownian motion (a complex oscillation) from any single random sequence. In this paper we show that the complex oscillation is computable in α and any number c for which $K(\alpha_{1:n}) > n - c$ for all n. In the terminology of Mathieu Hoyrup and Cristóbal Rojas, x_α is layerwise computable. We show that local minima are also computable and examine the computability of positions of local minima from this point of view.

Generic Case Decision Problems

Rod Downey
Victoria University, Wellington, New Zealand

Classical complexity using things like P, NP etc. seems often the wrong model for actual behaviour of problems. For example, consider the Simplex Algorithm, Polynomial Identity Testing and the like. Other models such as Parameterized complexity (Downey-Fellows), average case complexity (Gurevich-Levin), smoothed analysis (Spielman-modern version of average case) may or may not apply, the latter two being very difficult to work with as you need to find the distributions. In 2003, a new method suggested by Kapovich, Miasnikov, Schupp and Shpilrain mainly in the context of group theory. They called it *generic case complexity*.

The idea is that we have some kind of algorithm that works on a set of (Borel) density 1, and is *never wrong*, but could be partial. For example, any finitely presented group, and likely finitely generated group you fall over will be have a linear time generically decidable word problem.

This methodology presents unique problems to the computability theorist. I will survey the recent work of Jockusch and Schupp on this and give some new results from work jointly with those authors. This includes a new characterization of low c.e. sets using density and computability.

A data-type for complex Lipschitz functions

Abbas Edalat (Invited Speaker)
Imperial College London, UK

Complex Lipschitz functions, which include analytical and anti-analytical maps and have a well-behaved rectangular Lipschitz derivative, form a suitable class of functions for computation with good closure properties. We generalize the rectangular Lipschitz derivative of complex functions to a non-empty, convex and compact valued Scott continuous operator and show that it gives a new generalization of Cauchy Riemann equations for analytical maps. We then present a domain of computation of complex Lipschtiz maps which provides a framework to identify the computable complex Lipschiz maps and in particular the computable analytical or anti-analytical maps as different subsets of its maximal elements. We discuss extensions to higher dimensions.

On countable total orders which are Martin-Löf random

Willem Fouché (Invited Speaker)
Unisa, Pretoria, South Africa

By using structural Ramsey theory, we apply model-theoretic techniques to construct closed amenable subgroups G of the symmetric group S_∞ of a countable set, where S_∞ has the topology of pointwise convergence. We show how one can use ideas from Kechris, Pestov, Todorcevic and also from Glasner and Weiss in topological dynamics to effectively construct invariant means for the universal minimal flows associated with these amenable subgroups G. We study the Martin-Löf random objects in these universal minimal flows relative to the constructed invariant means. In this way we wish to identify what a Martin-Löf random total order on a countable set might be and relate such a total order to a description in terms of Kolmogorov complexity.

Strengthening difference randomness

Johanna Franklin
Dartmouth College, USA
(joint work with Keng Meng Ng)

In a previous paper, we defined difference randomness using n-r.e. tests. Here, we extend this notion to that of ω-r.e. randomness using f-r.e. tests for arbitrary recursive functions f. In this case, the n^{th} component of the test is an $f(n)$-r.e. set of open sets. Naturally, this lets us define f-r.e. randomness for a given f as well. Here, we present some basic results on ω-r.e. and f-r.e. randomness and describe their relationships to other strong randomness notions.

Limits on the Computational Power of Random Strings

Luke Friedman
Rutgers University, USA
(joint work with Eric Allender and William Gasarch)

How powerful is the set of random strings?

Letting R be the set of Kolmogorov-random strings, we ask what one can conclude about a computable set A, given that it is efficiently reducible to R. We present the first *upper bound* on the class of computable sets in P^R and NP^R.

The two most widely-studied notions of Kolmogorov complexity are the "plain" complexity $C(x)$ and "prefix" complexity $K(x)$; this gives rise to two common ways to define the set of random strings "R": R_C and R_K. (Of course, each different choice of universal Turing machine U in the definition of C and K yields another variant R_{C_U} or R_{K_U}.) Previous work on the power of "R" (for *any* of these variants has shown

- BPP $\subseteq \{A : A \leq_{tt}^{p} R\}$.
- PSPACE $\subseteq$ P^{R}.
- NEXP $\subseteq$ NP^{R}.

Since these inclusions hold irrespective of low-level details of how "R" is defined, we have e.g.: NEXP $\subseteq$ REC $\cap \bigcap_U \mathrm{NP}^{R_{K_U}}$. (Here, "REC" denotes the class of computable sets.)

Our main contribution is to present the *first* upper bounds on the complexity of sets that are efficiently reducible to R_{K_U}. We show:

- BPP $\subseteq$ REC $\cap \bigcap_U \{A : A \leq_{tt}^{p} R_{K_U}\} \subseteq$ PSPACE.
- NEXP $\subseteq$ REC $\cap \bigcap_U \mathrm{NP}^{R_{K_U}} \subseteq$ EXPSPACE.

Hence, in particular, PSPACE is sandwiched between the class of problems Turing- and truth-table-reducible to R.

As a side-product, we obtain new insight into the limits of techniques for derandomization from uniform hardness assumptions.

A General BSS Model over Arbitrary Structures

Christine Gassner
University Greifswald, Germany

L. Blum, M. Shub, and S. Smale created the fundamentals for a uniform complexity theory over the reals by introducing a real Turing machine which is able to process inputs of any length. We know that many non-uniform results could be transferred into the new model. However, on the other hand, the uniform treatment of inputs implies specific features which are typical for Turing machines and independent of the properties of the processed objects. A general BSS model and logical methods of investigation can help us to better understand the difference between the existing BSS und real RAM models.

Let $\mathbb{K} = (U; (f_j)_{j \in L}; (R_j)_{j \in M})$ be a structure where each f_j is an operation on U (of arity $n_j \geq 0$) and R_j is a relation on U (of arity m_j). The defined uniform model of computation over $\mathbb{K}$ involves many known models of computation. For the structure $(\{0,1\}; 0, 1; =)$, our model is closely related to Turing machines, and for structures over the reals, to the BSS model in [Blum et al. 89], to the linear, additive, and scalar models in [Meer 93], [Koiran 94], and [Gaßner 97] and to the arithmetical RAMs in [Reischuk 99]. The class of the underlying structures can also be restricted such that the power of our model corresponds to the model in [Hemmerling 95] and is closely related to a model in [Poizat 95]. Since we allow to use the equality (or identity) relation in tests only if this relation belongs to the structure, there are also common features with the feasible real RAMs defined and investigated in [Brattka Hertling 98].

We generalize the known infinite variant of the BSS model in order to treat all inputs of any length in a uniform way. Every uniform $\mathbb{K}$-*machine* $\mathcal{M}$ is equipped with registers $Z_1, Z_2, \ldots$ for the elements of U and with a fixed number of registers $I_1, I_2, \ldots, I_{k_\mathcal{M}}$ for indices in $\mathbb{N}^+ = \mathbb{N} \setminus \{0\}$ such that any input $(x_1, \ldots, x_n) \in U^\infty =_{\mathrm{df}} \bigcup_{i=1}^{\infty} U^i$ can be assigned to the first n registers $Z_1, \ldots, Z_n$. In addition to such an assignment (and in contrast to other real machines also having an infinite number of registers) the length n is stored in an index register before the computation starts. After the input the machine executes its *program* defined by a finite sequence of labeled instructions of the usual types until an output instruction is reached. Each operation and each test can be performed in a single step. The *computation*, *copy*, and *branching* instructions are defined as usual. The instructions $I_j := 1$, $I_j := I_j + 1$, and *if* $I_j = I_k$ *then goto* l_1 *else goto* l_2 allow to copy the content of any register by $Z_{I_s} := Z_{I_r}$. Since the size n of an input $(x_1, \ldots, x_n) \in U^\infty$ is also important for the non-deterministic acceptance and the uniform reducibility of a problem to another problem (cf. [Cucker Matamala 96]) we introduce *non-deterministic* machines which are able to guess an arbitrary number of elements $y_1, \ldots, y_m \in U$ in one step after the input.

Inspired by a talk by M. Ziegler and a remark by A. Pauly we also compare the BSS model with some real RAM models used in the Computational Geometry (cf. [Preparata Shamos 85]). Geometrical problems are very complex. In order to process n points of a d-dimensional space, it is natural to encode the entities including the number n and/or the number d. For arbitrary inputs in $\mathbb{R}^\infty$ it is not possible to compute the length of all theses inputs. Thus these RAMs work correctly only if the inputs are suitable codes of the considered objects. The models considered in [Grädel Meer 96] and in several other papers are rather related to this variant of the real RAM.

We discuss several computation problems, reducibility relations, proofs of the undecidability of halting problems and the existence of universal machines for some of the models of computation mentioned above. We explain why, in contrast to the full BSS model, many RAM models do not allow to use the known diagonalization argument for proving the undecidability of the corresponding halting problem or reduce all semi-decidable problems to the halting problem.

I thank the referees for their comments.

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Demuth randomness and strong jump-traceability

Noam Greenberg (Invited Speaker)
Victoria University of Wellington, New Zealand
(joint work with Dan Turetsky)

Kučera's programme concerns the relationship between incomplete random sets and the computably enumerable sets that they compute; the classic result being that every incomplete Δ^0_2 ML-random set computes a promptly simple c.e. set. An elaboration is the Hirschfeldt-Nies-Stephan result that every c.e. set computable from an incomplete ML-random set is K-trivial, whose converse, the incomplete random covering question, remains a major open problem in the field of algorithmic randomness. In general, the programme sets out to ask, what kind of random sets compute what kind of c.e. sets? A particularly pleasing situation is when classes of c.e. degrees can be characterised using randomness. For example, a set A is K-trivial if and only if it is computable from an A-random set.

When viewed in this light, the collection of c.e. *strongly jump-traceable* sets of Figueira-Nies-Stephan behaves particularly well, admitting sometimes surprising charcaterisations using randomness. For example, Hirschfeldt, Nies and Greenberg showed that a c.e. set is strongly jump-traceable if and only if it is computable from all ω-computably approximable random sets. Recently, Kučera and Nies showed that every c.e. set computable from a Demuth random set is strongly jump-traceable. We discuss whether this result could be turned into a new charcaterisation of strongly jump-traceable c.e. sets, and relate this to the investigation of the notion of lowness for Demuth randomness.

Traceable sets

Rupert Hölzl
Universität Heidelberg, Germany
(joint work with Wolfgang Merkle)

We investigate systematically into the various possible notions of traceable sets and the relations they bear to each other and to other notions such as diagonally noncomputable sets or complex and autocomplex sets. We review known notions and results that appear in the literature in different contexts, put them into perspective and provide simplified or at least more direct proofs.

In addition, we introduce notions of traceability and complexity such as infinitely often versions of jump traceability and of complexity, and derive results about these notions that partially can be viewed as a natural completion of the previously known results.

Finally, we give a result about polynomial-time bounded notions of traceability and complexity that shows that in principle the equivalences derived so far can be transferred to the time-bounded setting.

Computable analysis and algorithmic randomness

Mathieu Hoyrup (Invited Speaker)
INRIA Nancy, France
(joint work with Cristóbal Rojas)

Computable analysis and the algorithmic theory of randomness are expected to have strong connexions, as they both mix mathematical analysis and computability theory. Nevertheless those connexions are not much developed yet.

I will present some of them and show how the existence of a universal randomness test and the notion of randomness deficiency are deeply involved. In many situations, effectivizations of objects or properties from analysis correspond exactly to properties of random elements, effective in their randomness deficiencies. This correspondance is also quantitative. I will talk more precisely about absolute continuity, the Radon-Nikodym derivative, tightness of probability measures, the ergodic decomposition.

Negative results of computable analysis disappear if we restrict ourselves to random (or, more generally, typical) inputs

Vladik Kreinovich
University of Texas at El Paso, USA

It is well known that many operations are, in general, not computable: e.g., it is not possible to algorithmically decide whether two computable real numbers are equal, and it is not possible to compute the roots of a computable function. We propose to restrict such operations to certain “sets of typical elements” or “sets of random elements”.

In our previous papers, we proposed (and analyzed) physics-motivated definitions for these notions. In short, a set T is a *set of typical elements* if for every definable sequences of sets A_n with $A_n \supseteq A_{n+1}$ and $\bigcap_n A_n = \emptyset$, there exists an N for which $A_N \cap T = \emptyset$; the definition of a *set of random elements* with respect to a probability measure P is similar, with the condition $\bigcap_n A_n = \emptyset$ replaced by a more general condition $\lim_n P(A_n) = 0$.

In this talk, we show that if we restrict computations to such typical or random elements, then problems which are non-computable in the general case – like comparing real numbers or finding the roots of a computable function – become computable.

Solovay functions

Wolfgang Merkle (Invited Speaker)
Ruprecht-Karls-Universität Heidelberg, Germany
(joint work with Laurent Bienvenu, Rod Downey and André Nies)

Let a Solovay functions be any upper bound for prefixfree Kolmogorov complexity that is infinitely often tight up to an additive constant. Already in the 1970s, Solovay was able to construct a Solovay function that is computable. By a similar argument it can be seen that any reasonable time-bounded version of prefixfree Kolmogorov complexity is a computable Solovay function.

More recently, it has become clear that Solovay functions that are computable or, more generally, are right-c.e, i.e., can be effectively approximated from above, behave in many respects similar to prefixfree Kolmogorov complexity, which is itself a special case of such a function. What is more, some properties of prefixfree Kolmogorov complexity are not only shared by all right-c.e. Solovay functions, but are shared, among all right-c.e. functions, exactly by the Solovay functions. For example, by definition a sequence A is K-trivial if and only if for some constant c and all n, it holds that $K(A(0)\dots A(n-1)) \leq \mathrm{K}(n) + c$. Then the defined class is not changed by replacing in the upper bound the term $\mathrm{K}(n)$ by $g(n)$ for some right-c.e. Solovay function g. Moreover, among all right-c.e functions g, exactly for a Solovay function g such a replacement will again define the class of K-trivial sequences, whereas for all other right-c.e. functions the defined class will be either empty or uncountable.

Algorithmic randomness over general spaces

Kenshi Miyabe
Kyoto University, Japan

Algorithmic randomness over general spaces has been considered such as a computable topological space and a computable metric space. In this talk we investigate algorithmic randomness via three approaches on a computable topological space. First we define computable measures on a computable topological space and study computability of the evaluation. Next we define randomnesses via three approaches. Measure randomness is defined by tests and has a characterization by martingales. Complexity randomness is define by complexity and is equivalent to measure randomness under two natural conditions. Furthermore van Lambalgen's theorem holds when these randomnesses coincide.

Fine Computability of Probability Distribution Functions and Computability of Probability Distributions on the Real Line

Takakazu Mori
Kyoto Sangyo University, Japan
(joint work with Mariko Yasugi and Yoshiki Tsujii)

We extend our previous investigations on computability problems of probability distributions on the real line to the case where the corresponding probability distribution functions are Fine continuous.

Our main result can be stated as follows:

For an effectively Fine continuous sequence of probability distribution functions, its sequential computability is equivalent to the computability of the corresponding sequence of distributions.

We can also show an effective version of equivalence between the convergence of a sequence of probability distributions and that of the corresponding sequence of distribution functions.

Group-theoretic structure of zero sets of complex oscillations

Safari Mukeru
Department of Decision Sciences, UNISA, South Africa

A continuous function x on the unit interval is an algorithmically random Brownian motion, also known as a complex oscillation, when every probabilistic event A which holds almost surely with respect to the Wiener measure, is reflected in x, provided A has a suitably effective description. By using recent results in Fourier analysis, we shall explore the group-theoretic structure of the zero set of an algorithmically random Brownian motion.

Key words: algorithmic randomness, Kolgomorov complexity, Brownian motion, Fourier analysis, Fractal geometry, Effective definability.

Randomness and computable analysis: results and open questions

André Nies (Invited Speaker)
University of Auckland, New Zealand

The Lebesgue Differentiation Theorem is a theorem in classical analysis which states that a function of bounded variation is differentiable almost everywhere. In [1] we study effective versions of this theorem. Firstly, we show that a real z is computably random if and only if each nondecreasing computable function on the unit interval is differentiable at z. Next, we reprove a result already present in the work of Demuth (1975) (where it was phrased in constructivist language): z is Martin-Loef random if and only if each computable function of bounded variation is differentiable at z.

These results follow a very general scheme. One can vary the class of functions, the effectiveness notion, and one can replace differentiability by some other property that is classically known to hold almost everywhere for each function in the class. Our program, implicit already in the work of Demuth, is to characterize every major algorithmic randomness notion in this way. This gives new insight into the randomness notion: for instance we used the main result in [1] to show that computable randomness is base invariant. Such a characterization also clarifies the nature of the exception sets arising in the underlying classical theorem.

I will survey the status of this program, and pose several open questions. For example, J. Rute, extending work of N. Pathak, has announced that z is Schnorr random iff every function that is the indefinite integral of an L_1-computable function is differentiable at z. In [2] we show that differentiability of all computable Lipschitz functions characterizes computable randomness. In [2] we also describe the variation functions of computable Lipschitz functions.

[1] Brattka, Miller, and Nies. Randomness and differentiability. To appear.

[2] Freer, Kjos-Hanssen and Nies. Effective aspects of Lipschitz functions. To appear.

Randomness and K-triviality in computable metric spaces

André Nies
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(joint work with Alexander Melnikov)

We introduce and study the concept of K-triviality in computable metric spaces which is a generalization of the classical concept for sets. Recall that a set A is K-trivial if

$\forall n K(A \restriction_n) \leq K(n) + b$ for some constant b, where K is the prefix-free Kolmogorov complexity. In recent years K-trivials have become one of the central objects of study in algorithmic randomness. It is known that there is a c.e. example of a non-computable K-trivial set [1]. See [2] for more about K-trivial sets.

To study K-triviality in metric spaces, we first introduce the concept of K-triviality for functions. We say that a point x in a computable metric space is *K-trivial* if it has a Cauchy name that is K-trivial as a function. We show that K-triviality in a metric space is invariant under change of computable structure to an equivalent one. We demonstrate that every computable (non-empty) perfect Polish space contains a non-computable K-trivial point. Then we introduce the notion of the standard cost function $\hat{c}_{\mathcal{K}}(\epsilon, s)$ in the case of metric spaces. We conjecture that a point in a perfect Polish space with computable structure $\{\alpha_n\}_{n\in\mathbb{N}}$ is K-trivial if and only if there exists a computable sequence $g : \mathbb{N} \to \{\alpha_n\}_{n\in\mathbb{N}}$ such that: (1) $x = \lim_s g(s)$, and (2) g obeys $\hat{c}_{\mathcal{K}}(\epsilon, s)$. We are able to prove the "only if" direction, and for the "if" direction we show that the limit exists.

In contrast to K-triviality, we give the notion of incompressibility in approximation for metric spaces. This notion is equivalent to ML-randomnes in the case of the Cantor space. We show that incompressibility in approximation is invariant under a change of computable structure to an equivalent one.

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[2] André Nies. *Computability and randomness*, volume 51 of *Oxford Logic Guides*. Oxford University Press, Oxford, 2009.

The degree structure of Weihrauch reducibility

Arno Pauly
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(joint work with Kojiro Higuchi)

We answer a question by Vasco Brattka and Guido Gherardi by proving that the Weihrauch-lattice is not a Brouwer algebra. The computable Weihrauch-lattice also is not a Heyting algebra, but the continuous Weihrauch-lattice is. We further investigate embeddings of the Medvedev-degrees into the Weihrauch-degrees.

Compactness and the Effectivity of Uniformization

Robert Rettinger
FernUniversität Hagen, Germany

We give new proofs of effective versions of the Riemann mapping theorem, its extension to multiply connected domains and the uniformization on Riemann surfaces. Astonishingly, in the presented proofs we need barely more than computational compactness and the classical results.

Using ideas of Kolmogorov complexity to study behavioural patterns in animals

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(joint work with Boris Ryabko)

One of the main problems of ethology and evolutionary biology is searching for a reliable criterion for evaluation of the individual behavioral variability in populations and communities. Complexity of behavioral patterns could serve as a good criterion for inter- and intraspecies comparison; however, there are no reliable tools for studying complexity of animal behavioral patterns. The prevalent method of comparative ethological studies is based on the analysis of the so-called ethograms, that is, recordings of behavioral sequences as alphabets consisting, in average, of 10–15 symbols or letters each corresponding to a certain behavioral unit (an act). Attempts to apply the probabilistic approach for description and comparison of animal behaviors meet methodological difficulties. Measures of evaluation of complexity of behaviors based on the concept of Kolmogorov complexity should be more adequate.

We suggest a method for evaluating the complexity of animal behavioral patterns based on the notion of Kolmogorov complexity, with ants' hunting behavior as an example. We compared complete (successful) and incomplete hunting stereotypes in members of a natural ant colony and in naive laboratory-reared ants. We represent behavioral sequences as "texts", and compress them using a data compressor. Behavioral units (10 in total), singled out from video records and denoted by letters, served as an alphabet. Successful hunting stereotypes appeared to be characterized by smaller complexity than incomplete ones. A few naive "born hunters" which enjoy "at once and entirely" complete hunting stereotypes are characterized by a lower level of complexity of hunting behavior. We conclude that innate complete stereotypes have less redundancy and are more predictable, and thus less complex. This gives an important tool to ethologists for comparative analysis of individual behavioral variability and for extracting "source" (completely innate) behavioral patterns by comparing behavioral sequences of different levels of complexity and not resorting to rearing naive animals. This is particularly important for evolutionary and ethological studies in the field.

Keywords: Kolmogorov complexity, data compression, animal behavior, variability, redundancy, predictability, ants.

Perfect Steganography and Kolmogorov Complexity

Daniil Ryabko
INRIA Lille
(joint work with Boris Ryabko)

The contribution of the presented work is two-fold. First, we describe a perfectly secure stegosystem for the case of i.i.d. (and finite-memory) sources of covertexts. Second, we use Kolmogorov complexity arguments to show that there are sources of covertexts (that do not obey the finite memory assumption) for which perfectly secure steganographic systems must have the size exponential in the size of the message to transmit.

Two New degrees based on Enumeration Orders and their Related Equivalency Relations

Ali Akbar Safilian
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(joint work with Farzad Didehvar)

In this paper we define two novel degrees in r.e. sets, weaker than Turing reducibility and study their related equivalency relations.
Every recursive set can be ordered in a computable way increasingly. Also, any r.e. set which can be ordered increasingly is a recursive set. Therefore, increasing order type has a very tight relation to recursive sets. Can we say some similar things about non-recursive r.e. sets? Is there any relation between some special order types based on natural enumeration and r.e. sets?
In the investigation based on the above description, first we have defined a reducibility named "Enumeration Order Reducibility" based on enumeration orders. Later on, an equivalence relation among listings of different r.e. sets and also an equivalence relation among r.e. sets are defined based on Enumeration Order Reducibility and some of their properties are studied.
Subsequently, another concept named "type-2 Enumeration Order Reducibility" is introduced and its related equivalency relation are studied as well.

Keywords: Turing machine, Listing, Enumeration order reducibility, Enumeration order equivalency, Type-2 enumeration order equivalency.

The Coincidence Problem in Computable Analysis

Matthias Schröder (Invited Speaker)
Universität der Bundeswehr, Munich, Germany

In functional programming, there are essentially two approaches to computability on the real numbers. The *extensional* approach assumes an idealistic functional language containing the real numbers as an own datatype. The *intensional* approach uses data structures of ordinary functional languages and encodes real numbers as streams using the signed-digit representation.

It is known that both approaches yield the same classes of Type-1 and Type-2 functionals over the reals. However, whether this is also the case for functionals of type $n \geq 3$ was an open problem for a long time. Bauer, Escardó and Simpson [1] were the first to discover that there is a relationship to topological properties of the Kleene-Kreisel continuous functionals over the natural numbers $\mathbb{N}$. Normann [2] came up with a purely topological condition on the Kleene-Kreisel spaces that is equivalent to the Coincidence Problem.

In this talk, I will prove the non-coincidence of the hierarchies by showing that the Kleene-Kreisel space $\mathbb{N}^{(\mathbb{N}^{\mathbb{N}})}$ does not satisfy Normann's equivalence condition. By Normann's result the two hierarchies do not agree from level 3 on. This is the first previously unknown level. I will also present an example of a Type-3 functional over the reals that is "extensional", but not "intensional".

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Proofs by Induction

Peter Schuster (Invited Speaker)
University of Leeds, England, and Universität München, Germany
(based on joint work with Ulrich Berger and on communications by Thierry Coquand)

Many a concrete theorem of abstract mathematics admits a short and elegant proof by contradiction but with the so-called Lemma of Kuratowski-Zorn (ZL). A few of these theorems have recently turned out to follow in a direct and elementary way from the Principle of Open Induction coined by Raoult. Typically a proof of the latter kind is of a more algorithmic character, and may be obtained systematically from a proof of the former sort. The ideal objects characteristic of any invocation of ZL are eliminated, and it is made possible to pass from classical to intuitionistic logic. This approach is intended as a contribution to a partial realization of Hilberts Programme, and was motivated by related work of Berger, Coquand and by the rise of dynamical and formal methods in algebra.

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Symbolic dynamics: entropy = dimension = complexity

Stephen G. Simpson (Invited Speaker)
Pennsylvania State University, USA

We prove some fundamental results in symbolic dynamics. Let G be a countable semigroup, specifically $G = (\mathbb{N}^d, +)$ or $G = (\mathbb{Z}^d, +)$ where d is a positive integer. Let A be a finite set of symbols. A *G-subshift* is a nonempty set $X \subseteq A^G$ which is G-invariant and topologically closed. Note that we impose no finiteness or computability hypothesis on X. We prove that, with respect to the standard metric on X, the Hausdorff dimension of X coincides with the effective Hausdorff dimension of X and with the topological entropy of X. We obtain a sharp characterization of the Hausdorff dimension of X in terms of the Kolmogorov complexity of the finite configurations of symbols which occur in X.

Structures Reflecting Recursion Theoretic Properties

Theodore A. Slaman (Invited Speaker)
University of California, Berkeley
(joint work with Noam Greenberg and Antonio Montalbán)

We will discuss some results obtained pertaining to the question "Which recursion theoretic properties of a real number X are indicated by the isomorphism types of the structures that X recursively represents?" We will show that there is a structure M such that X's being non-hyperarithmetic is equivalent to X's recursively representing M. We will also exhibit structures that distinguish genericity from randomness.

Constructive Notions of Compactness in Apartness Spaces

Thomas Steinke
University of Canterbury, New Zealand

We extend the constructive theory of apartness spaces developed by Bridges and Vîţă. In particular, we propose new notions of compactness and show how they relate to existing notions thereof. Our proposed criteria make use of apartness between sets and, hence, the extra structure offered by an apartness space. Previous criteria were purely toplogical in nature. Our criteria are based on the (classical) observation that, in a compact space, two disjoint closed sets are necessarily apart.

We give three definitions of varying strength. These can be related to existing compactness criteria such as total boundedness, the anti-Specker property, and Diener's neat compactness. Moreover, we are able to demonstrate that the strongest two of our criteria are classically equivalent to open-cover compactness in the context of a separable metric space. The weakest of our criteria can be characterised in terms of a constructive analogue to the uniform continuity theorem. These definitions are promising and lead to many further questions.

A Chaitin Ω number based on compressible strings

Kohtaro Tadaki
Chuo University, Japan

In 1975 Chaitin introduced his Ω number as a concrete example of random real. The real Ω is defined based on the set of all halting inputs for an optimal prefix-free machine U, which is a universal decoding algorithm used to define the notion of program-size complexity $H(s)$ for a finite binary string s. Chaitin showed Ω to be random by discovering the property that the first n bits of the base-two expansion of Ω solve the halting problem of U for all binary inputs of length at most n. In this talk we introduce a new variant Θ

of Chaitin Ω number, based on compressible strings, i.e., finite binary strings s such that $H(s) < |s|$. The real Θ is defined as $\sum_s 2^{-|s|}$ where the sum is over all compressible strings s. The first n bits of the base-two expansion of Θ enable us to calculate a random finite string of length n, i.e., a finite binary string s such that $|s| = n$ and $|s| \leq H(s)$. Based on this property, we show Θ to be random. In addition, we generalize Θ to $\Theta(Q, R)$ with reals $Q, R > 0$ by $\Theta(Q, R) = \sum_s 2^{-|s|/Q}$ where the sum is over all finite binary strings s such that $H(s) < R\,|s|$. We then study its randomness properties. In particular, we show that the computability of the real $\Theta(T, 1)$ gives a sufficient condition for a real $T \in (0, 1)$ to be a fixed point for partial randomness, i.e., to satisfy the condition that the compression rate of T equals to T.

Properties of fibers of optimal prefix-free machines

Kohtaro Tadaki
Chuo University, Japan

An optimal prefix-free machine U is a universal decoding algorithm used to define the notion of program-size complexity $H(s)$ for a finite binary string s. In this talk we investigate the properties of the fiber of U at a finite binary string s, i.e., the set $U^{-1}(s) = \{\, p \mid U(p) = s \,\}$. We first investigate the number of programs in $U^{-1}(s)$, and show the following: (i) While keeping $H(s)$ unchanged for all s, we can modify U so that each $U^{-1}(s)$ is a finite set, where the number of programs in $U^{-1}(s)$ is bounded to the above by some total recursive function $f(s)$. (ii) This upper bound $f(s)$ cannot be chosen to be tight at all. (iii) As a result, even in the case where all $U^{-1}(s)$ are a finite set, the number of programs in $U^{-1}(s)$ is not bounded to the above on all finite binary strings s. (iv) While keeping $H(s)$ unchanged for all s, we can modify U so that each $U^{-1}(s)$ is an infinite set. We next investigate the distribution of programs in $U^{-1}(s)$. We estimate the distribution using the program-size complexity $H(s)$, and then show that the estimation is tight while introducing the notion of maximally redundant prefix-free machine.

Enumerating Randoms

Jason Teutsch
Ruprecht-Karls-Universität Heidelberg
(joint work with Bjørn Kjos-Hanssen and Frank Stephan)

This talk investigates enumerability properties for classes of random reals which permit recursive approximations from below, or *left-r.e.* reals. In addition to pinpointing the complexity of left-r.e. Martin-Löf, computable, Schnorr, and Kurtz random reals and their complementary classes, we find that there exist uniform characterizations of the third and fourth levels of the arithmetic hierarchy purely in terms of these notions. More generally, there exists an equivalence between arithmetic complexity and existence of numberings for classes of left-r.e. reals with *shift-persistent elements*. We also remark that while some classes (e.g. Martin-Löf randoms and Kurtz non-randoms) have left-r.e. numberings, there is no canonical, or *acceptable*, left-r.e. numbering for any class of left-r.e. randoms.

The Radon-Nikodym operator is not computable

Klaus Weihrauch
University of Hagen, Germany
(joint work with Mathieu Hoyrup and Cristobal Rojas)

Let λ be the Lebesgue measure on the Borel sets of the unit interval. By the Radon-Nokodym theorem, for every measure ν on the Borel sets such that $\nu \leq \lambda$ there is an (unique) λ-integrable function h such that $\nu(A) = \int h\, d\lambda$. Consider canonical representations of the bounded measures on the Borel sets and of the λ-integrable functions. Then the Radon-Nikodym operator $\mathrm{RN} : \nu \mapsto h$ is not computable. Even more, let EC be the operator transforming every enumeration of a set of natural numbers to its characteristic function (w.r.t. canonical representations), which is not computable. Then $\mathrm{EC} \equiv_{\mathrm{W}} \mathrm{RN}$, that is, one application of EC allows to compute RN and one application or RN allows to compute EC.

Thurston equivalence and algorithmic decidability

Michael Yampolsky (Invited Speaker)
University of Toronto, Canada

Suppose f is a covering map of the 2-sphere $f : S^2 \to S^2$ of a finite degree $d > 1$ and having a finite number of branch points. Assume further that the orbit of every branch point is finite. Such a *postcritically finite* map can be described by a finite amount of combinatorial information up to a *Thurston equivalence*. A celebrated theorem of Thurston answers the following question: can f be realized analytically as a rational mapping R of the Riemann sphere $\hat{\mathbb{C}}$, and, if yes, is R unique up to a normalization? This theorem, and the tools its proof provides, have been central to the development of the field of Complex Dynamics. It is often practically important to compute the coefficients of R given a combinatorial description of f.

I will present some recent works on algorithmic decidability of Thurston equivalence. Jointly with S. Bonnot and M. Braverman we have shown that the question of equivalence to a rational map is algorithmically decidable. The general equivalence problem between pairs of postcritically finite branched coverings of S^2 is much more challenging. I will describe an approach to its solution.

Relative Computability and Uniform Continuity of Relations

Martin Ziegler
Technische Universität Darmstadt, Germany

A type-2 computable real function is necessarily continuous; and this remains true for relative, i.e. oracle-based computations. Conversely, by the Weierstrass Approximation Theorem, every continuous $f : [0, 1] \to \mathbb{R}$ is computable relative to some oracle. In their search for a similar topological characterization of relatively computable *multi*-valued functions $f : [0, 1] \rightrightarrows \mathbb{R}$ (aka relations), Brattka and Hertling (1994) have considered two notions: weak continuity (which is weaker than relative computability) and strong continuity (which is stronger than relative computability). Observing that *uniform* continuity plays a crucial role in the Weierstrass Theorem, we propose and compare several notions of uniform continuity for relations. Here, due to the additional quantification over values $y \in f(x)$, new ways of ordering quantifiers arise, yet none of them turn out as satisfactory. We are then led to a notion of uniform continuity based on the Henkin quantifier; and prove it necessary for relative computability.

Combinatorial characterizations of extractors and Kolmogorov extractors

Marius Zimand
Towson University, USA

Randomness extraction is the process of constructing a source of randomness of high quality from one or several sources of randomness of lower quality. The problem can be modeled using probability distributions and min-entropy to measure their quality and also by using individual strings and Kolmogorov complexity to measure their quality.

A function f achieving the objective in the first setting is called an *extractor*. Extractors have been instrumental in obtaining important results in derandomization, cryptography, data structures, and other areas. *Kolmogorov extractors* are the functions f achieving the objectives in the second setting and they have applications in Kolmogorov complexity theory and in algorithmical randomness (for example, see Chapter 12 in the Downey and Hirschfeldt monograph.).

We present characterizations of extractors and Kolmogorov extractors in terms of a combinatorial object called balanced table. These characterizations provide an alternative proof for the relation between extractors and Kolmogorov extractors, first obtained by Fortnow, Hitchcock, A. Pavan, Vinodchandran and Wang [ICALP 2006] and by Hitchcock, A. Pavan and Vinodchandran [FSTTCS 2009].