Effective refinements of classical theorems in descriptive set theory

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Recursive Polish spaces

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Definition

Suppose that $(\mathcal{X}, d)$ is a separable complete metric space. A **recursive presentation** of $(\mathcal{X}, d)$ is a function $\mathbf{r} : \omega \to \mathcal{X}$ such that

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1. the set $\{\mathbf{r}_n \mid n \in \omega\}$ is dense in $\mathcal{X}$,
2. the relations $P_<, P_\leq \subseteq \omega^3$ defined by

$$P_<(i,j,s) \iff d(\mathbf{r}_i, \mathbf{r}_j) < q_s$$
$$P_\leq(i,j,s) \iff d(\mathbf{r}_i, \mathbf{r}_j) \leq q_s$$

   are recursive.

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A Polish space $\mathcal{X}$ is a **recursive Polish space** if there exists a pair $(d, \mathbf{r})$ as above.

We encode the set of all finite sequences of naturals by a natural in a recursive way and we denote the corresponding set by Seq.

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2. *Every zero-dimensional Polish space is homeomorphic to a closed subset of $\mathcal{N}$.*

Suslin schemes

Definition

A **Suslin scheme** on a Polish space $\mathcal{X}$ is a family $(A_s)_{s \in \mathrm{Seq}}$ of subsets of $\mathcal{X}$ indexed by Seq.

We say that $(A_s)_{s \in \mathrm{Seq}}$ is of **vanishing diameter** if for all $\alpha \in \mathcal{N}$ we have that

$$\lim_{n \to \infty} \mathrm{diam}(A_{\overline{\alpha}(n)}) = 0,$$

for some compatible distance function d, where $\overline{\alpha}(n)$ is the code of the finite sequence $(\alpha(0), \ldots, \alpha(n-1))$.

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For every Suslin scheme $(A_s)_{s \in \mathrm{Seq}}$ on a Polish space $\mathcal{X}$ of vanishing diameter we assign the set

$$D = \{\alpha \in \mathcal{N} \mid \cap_{n \in \omega} A_{\overline{\alpha}(n)} \neq \emptyset\}.$$

Since the Suslin scheme is of vanishing diameter the intersection $\cap_{n \in \omega} A_{\overline{\alpha}(n)}$ is at most a singleton.

Definition

We define the partial function $f : \mathcal{N} \rightharpoonup \mathcal{X}$ by

$$f(\alpha) \downarrow \iff \alpha \in D$$
$$f(\alpha) \downarrow \implies f(\alpha) = \text{the unique } x \in \cap_{n \in \omega} A_{\overline{\alpha}(n)}.$$

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A Suslin scheme $(A_s)_{s\in\mathrm{Seq}}$ is **semirecursive** (**recursive**) if the set $A \subseteq \mathrm{Seq} \times \mathcal{X}$ defined by $A(s,x) \iff x \in A_s$, (so that the s-section of A is exactly the set A_s) is **semirecursive** (**recursive**).

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We notice that semirecursive Suslin schemes consist of open sets and that recursive Suslin schemes consist of clopen sets.

Lusin schemes

Definition

A **Lusin scheme** on a Polish space $\mathcal{X}$ is a Suslin scheme $(A_s)_{s \in \mathrm{Seq}}$ with the properties

1. $A_{s^\frown i} \cap A_{s^\frown j} = \emptyset$ for all $s \in \mathrm{Seq}$ and $i \neq j$, and
2. $A_{s^\frown i} \subseteq A_s$ for all $s \in \mathrm{Seq}$ and $i \in \omega$.

The notions of “vanishing diameter", “associated map" and “being semirecursive/recursive" apply also to Lusin schemes in the obvious way.

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3. if $(A_s)_{s \in \mathrm{Seq}}$ is a Lusin scheme and every A_s is open then f is a homeomorphism between D and $f[D]$,

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1. the associated map $f : D \to \mathcal{X}$ is continuous,
2. if every A_s is open and $A_s \subseteq \cup_i A_{s^\frown i}$ then f is open,
3. if $(A_s)_{s \in \mathrm{Seq}}$ is a Lusin scheme and every A_s is open then f is a homeomorphism between D and $f[D]$,
4. if $(A_s)_{s \in \mathrm{Seq}}$ is a Lusin scheme and every A_s is closed then D is closed as well.

Lemma

Suppose that $\mathcal{X}$ is recursive Polish space and that $(A_s)_{s \in \mathrm{Seq}}$ is a semirecusive Suslin scheme with associated map the function f and $\mathrm{diam}(A_s) < 2^{-\mathrm{lh}(s)}$ for all $s \in \mathrm{Seq}$ for some compatible pair $(d, \mathbf{r})$. Then the partial function

$$f : \mathcal{N} \rightharpoonup \mathcal{X}$$

is recursive on its domain.

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is recursive on its domain.

If moreover the family $(A_s)_{s \in \mathrm{Seq}}$ is a Lusin scheme then the inverse partial function

$$f^{-1} : \mathcal{X} \rightharpoonup \mathcal{N}$$

is recursive on its domain as well.

Lemma

For every recursive Polish space $\mathcal{X}$ and every compatible pair $(d, \mathbf{r})$ there exists a semirecursive Suslin scheme $(A_s)_{s \in \mathrm{Seq}}$ with the following properties.

1. Every A_s is non-empty,
2. $\mathrm{diam}(A_s) < 2^{-\mathrm{lh}(s)}$ for all $s \in \mathrm{Seq}$,
3. $A_0 = \mathcal{X}$,
4. $A_s = \cup_{i \in \omega} A_{s^\frown i} = \cup_{i \in \omega} \overline{A}_{s^\frown i}$.

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3. $A_\emptyset = \mathcal{X}$,
4. $A_s = \cup_{i \in \omega} A_{s^\frown i} = \cup_{i \in \omega} \overline{A}_{s^\frown i}$.

Basic tool for the proof.

There exists a recursive set $I \subseteq \omega^5$ such that for all $(n, i, k) \in \omega^3$ we have

$$B(\mathbf{r}_i, q_k) = \bigcup_{(j,m) \in I_{(n,i,k)}} B(\mathbf{r}_j, q_m) = \bigcup_{(j,m) \in I_{(n,i,k)}} \overline{B(\mathbf{r}_j, q_m)}$$

and $\mathrm{diam} B(\mathbf{r}_j, q_m) \leq 2^{-n+1}$ for all $(j, m) \in I_{(n,i,k)}$. ⊣

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Sketch of the proof.

We consider the semirecursive Suslin scheme $(A_s)_{s \in \mathrm{Seq}}$ of the preceding Lemma.

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We consider the semirecursive Suslin scheme $(A_s)_{s \in \mathrm{Seq}}$ of the preceding Lemma. The associated map f is a total, surjective and recursive.

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Sketch of the proof.

We consider the semirecursive Suslin scheme $(A_s)_{s \in \mathrm{Seq}}$ of the preceding Lemma. The associated map f is a total, surjective and recursive. Moreover since every A_s is open and $A_s \subseteq \cup_i A_{s \hat{} i}$ for all s, i it follows that f is open. $\dashv$

Definition

A recursive Polish space $\mathcal{X}$ is **recursively zero-dimensional** if there exists a compatible pair $(d, \mathbf{r})$ such that the relation $I \subseteq \mathcal{X} \times \omega \times \omega$ defined by

$$I(x, i, s) \iff d(x, \mathbf{r}_i) < q_s$$

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is recursive.

Lemma

For every recursively zero-dimensional Polish space $\mathcal{X}$ and every compatible pair $(d, \mathbf{r})$ for $\mathcal{X}$ there exists a recursive Lusin scheme $(A_s)_{s \in \mathrm{Seq}}$ with the following properties.

1. $A_0 = \mathcal{X}$,
2. $A_s = \cup_i A_{s^\frown i}$ and
3. $\mathrm{diam}(A_s) < 2^{-\mathrm{lh}(s)}$

for all $s \in \mathrm{Seq}$ and all $i \in \omega$.

Theorem

For every recursively zero-dimensional Polish space $\mathcal{X}$ there exists a recursive injection $g : \mathcal{X} \rightarrowtail \mathcal{N}$ such that the set $g[\mathcal{X}]$ is $\Pi^0_1(\varepsilon)$ for some $\varepsilon \in \Delta^0_2$.
Moreover the inverse function $g^{-1} : g[\mathcal{X}] \rightarrowtail\!\!\!\!\rightarrow \mathcal{X}$ is computed by a semirecursive subset of $\mathcal{N} \times \omega^2$ on $\mathcal{Y}$. In particular the inverse function g^{-1} is continuous.

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Sketch of the proof.

We consider the recursive Lusin scheme $(A_s)_{s \in \mathrm{Seq}}$ of the preceding Lemma and its associated map $f : D \to \mathcal{X}$.

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We consider the recursive Lusin scheme $(A_s)_{s \in \mathrm{Seq}}$ of the preceding Lemma and its associated map $f : D \to \mathcal{X}$. Then f is continuous and bijective.

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We consider the recursive Lusin scheme $(A_s)_{s \in \mathrm{Seq}}$ of the preceding Lemma and its associated map $f : D \to \mathcal{X}$. Then f is continuous and bijective. We take $g = f^{-1} : \mathcal{X} \rightarrowtail D$.

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Sketch of the proof.

We consider the recursive Lusin scheme $(A_s)_{s \in \mathrm{Seq}}$ of the preceding Lemma and its associated map $f : D \to \mathcal{X}$. Then f is continuous and bijective. We take $g = f^{-1} : \mathcal{X} \rightarrowtail D$. The set D is closed (see preceding slides) and in fact it is $\Pi^0_1(\varepsilon)$ for some $\varepsilon \in \Delta^0_2$. $\dashv$

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It holds that $g[\mathcal{X}] = D$ and

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We define $\varepsilon(s) = 1$ exactly when there exists i such that $\mathbf{r}_i \in A_s$ and 0 otherwise, so that

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Theorem (Kleene's Recursion Theorem)

For every partial function $f : \omega \rightharpoonup \omega$, which is recursive on its domain, there exists some e^* such that for all $n \in \text{Domain}(f)$

$$\{e^*\}(n)\downarrow \text{ and } f(n) = \{e^*\}(n).$$

Thank you for your attention!