Topological rays and lines as co-c.e. sets

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Computable metric spaces

Definition

A triple (X, d, α) is a **computable metric space** if (X, d) is a metric space and $\alpha : \mathbb{N} \to X$ is a sequence with a dense image in X such that the function $\mathbb{N}^2 \to \mathbb{R}$

$$(i, j) \mapsto d(\alpha_i, \alpha_j)$$

is computable.

The points $\alpha_0, \alpha_1, \ldots$ are **rational points** or **special points**.

Computable metric spaces

Definition

A point $x \in X$ is **computable** in (X, d, α) if there exists a computable function $f : \mathbb{N} \to \mathbb{N}$ such that

$$d(\alpha_{f(i)}, x) < 2^{-i}$$

for all $i \in \mathbb{N}$.

Effective enumerations

- A set I is a **rational ball** if $I = B(\lambda, \rho)$ where λ is a rational point and $\rho \in \mathbb{Q}^+$.
- We denote by (I_k) and $(\widehat{I_k})$ some fixed effective enumerations of open and closed rational balls respectively.

Co-c.e. sets

Definition

Let (X, d, α) be a computable metric space. A closed subset $S \subseteq X$ is a **co-computably enumerable set** if there exists a computable function $f : \mathbb{N} \to \mathbb{N}$ such that

$$X \setminus S = \bigcup_{i \in \mathbb{N}} I_{f(i)}$$

Computable sets

Definition

Let (X, d, α) be a computable metric space. A set $S \subseteq X$ is **computable** if

1. S is co-c.e.;
2. S is computably enumerable i.e. the set
$$\{i \in \mathbb{N} : S \cap I_i \neq \emptyset\}$$
is computably enumerable.

Computable sets

- If S is computable then it is clearly co-c.e.
- On the other hand if S is co-c.e., S doesn't have to be computable.

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Example

There exists a a co-c.e. line segment $[0, a]$ with uncomputable a.

Question

- Let (X, d, α) be a computable metric space. Let $S \subseteq X$.
- Which topological conditions we have to impose on S so that the implication

$$S \text{ co-c.e} \implies S \text{ computable}$$

  holds ?
- First we set our ambient space!

Nice computable metric spaces

Definition

A computable metric space (X, d, α) is **nice** if it has the effective covering property and compact closed balls.

Nice computable metric spaces

Remark

In any nice computable metric space (X, d, α) we can effectively enumerate all rational open sets which cover a given **compact** *co-c.e. set S.*

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Nice computable metric spaces

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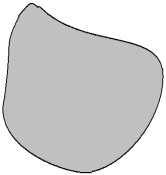
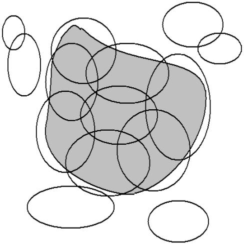
In any nice computable metric space (X, d, α) *we can effectively enumerate all rational open sets which cover a given* **compact** *co-c.e. set* S.

- We observe only nice computable metric spaces.
- What can we say about conditions under which a co-c.e. set S is computable in such an ambient space?

Nice computable metric spaces



Nice computable metric spaces



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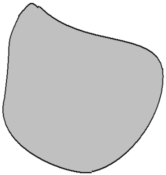

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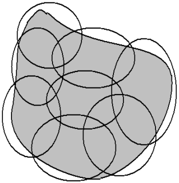

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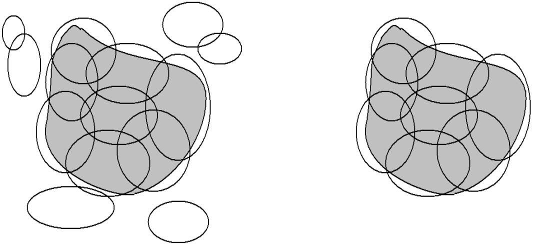

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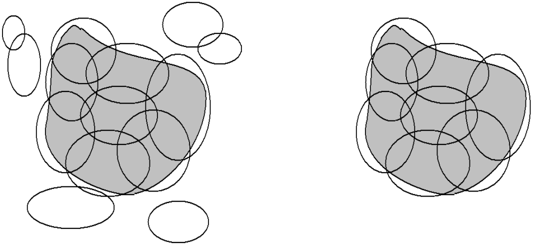
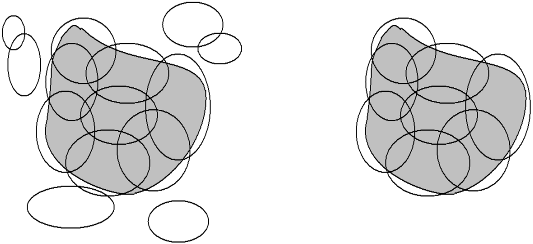


Remark

For compact co-c.e. sets the effective appoximation by a rational set implies computability!

Nice computable metric spaces



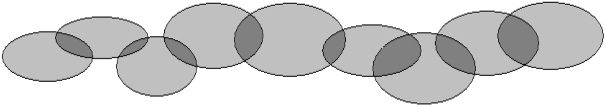
Problem

If a rational set $J = B_1 \cup \cdots \cup B_k$ covers S we cannot effectively determine which B_i intersect S.

Chains

Definition

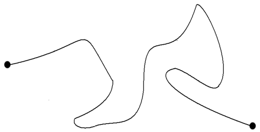
1. A finite sequence $\mathcal{C} = (C_0, \ldots, C_m)$ of open sets in X is a **chain** if
$$|i - j| > 1 \implies C_i \cap C_j = \emptyset$$
for all $i, j \in \{0, \ldots, m\}$. Each C_i is called a **link**.
2. For $\epsilon > 0$ a finite sequence $C_0, \ldots, C_m$ is an ϵ-**chain** if diametar of each C_i is less than ϵ.



Arcs

Definition

A metric space A is an **arc** if A is homeomorphic to the segment $[0, 1]$.



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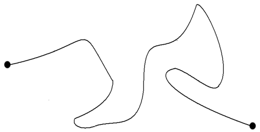


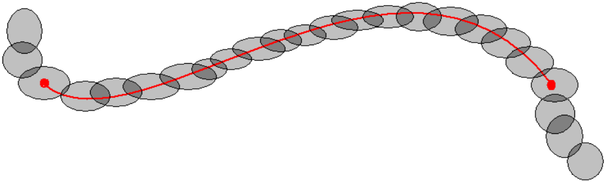
Remark

Every arc is a compact set.

Arcs

Lemma

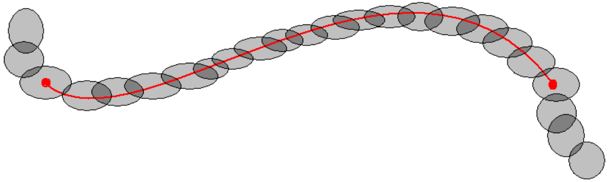
Let (X, d, α) be a nice computable metric space. Let $\epsilon > 0$. Let S be an arc in X. Then there exists an ϵ-chain which covers S. Furthermore, we can effectively find an ϵ-chain with rational links which covers S.



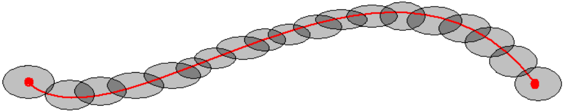
Arcs

Problem

We have "unnecessary" links which can not be effectively detected!



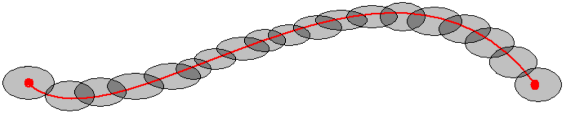
Arcs with computable endpoints



Remark

We can effectively enumerate all chains which start and end at the endpoints!

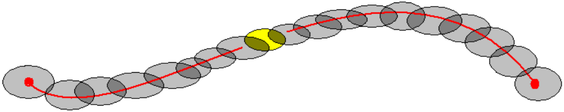
Arcs with computable endpoints



Remark

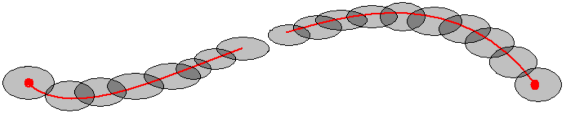
Each link of such a chain must intersect the arc!

Arcs with computable endpoints



Suppose there's a link that **does not** intersect the arc.

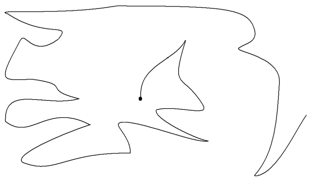
Arcs with computable endpoints



Contradiction!

Topological rays

- A metric space R is a **topological ray** if R is homeomorphic to the interval $[0, \infty\rangle$.



- If R is a topological ray and $f : [0, \infty\rangle \to R$ a homeomorphism. Then the point $f(0)$ is called the **endpoint** of R.

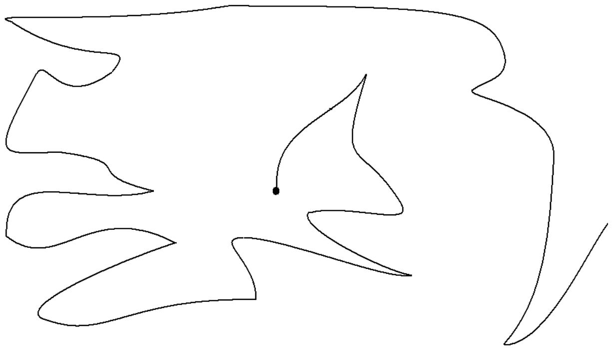
Remark

Being an endpoint doesn't depend on the choice of f.

Topological rays

- If we have a closed set which is a topological ray then "it's tail converges to infinity".
- If we drop the condition that R is closed then this is not true! (for example set R=$[0, 1\rangle$))

Closed topological rays ("tail converges to infinity")



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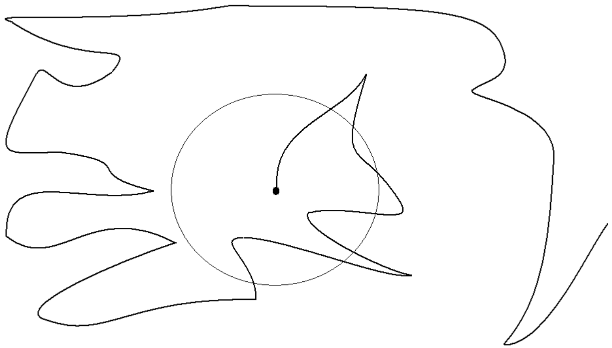

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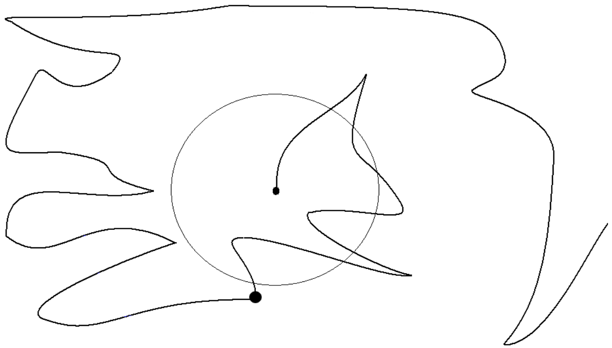

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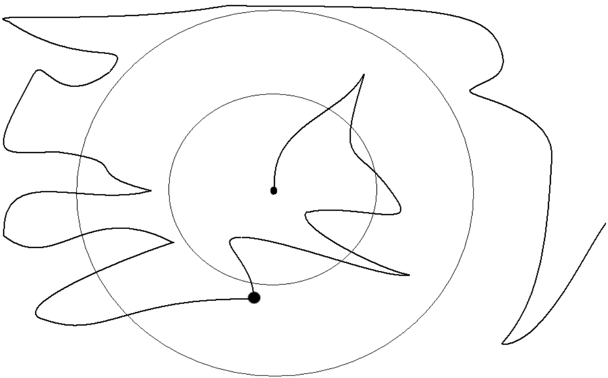

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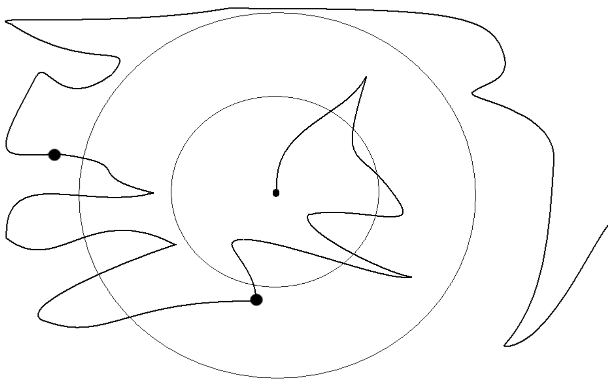

Problems

- A topological ray is **not compact**!

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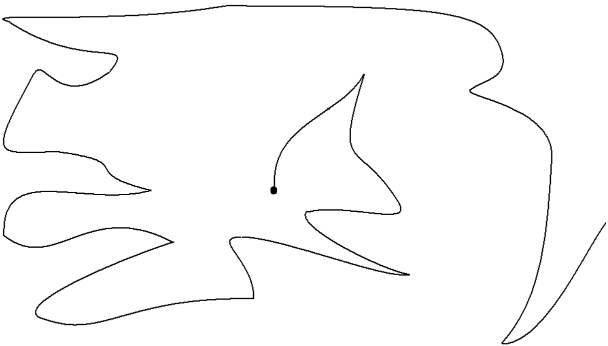
Nevertheless, we proved the following theorem.

Computability of co-c.e. topological rays

Theorem

Let (X, d, α) be a nice computable metric space. Let $R \subseteq X$ be a co-c.e. topological ray with a computable endpoint. Then R is computable.

Proof(sketch)



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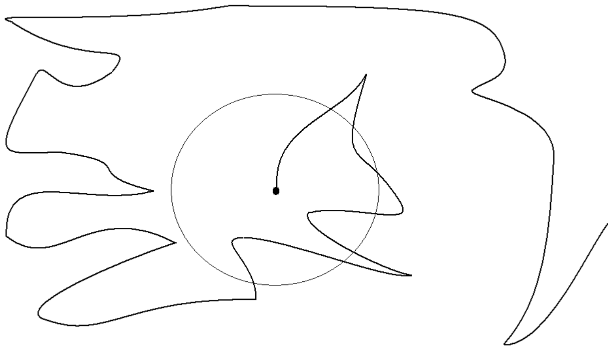

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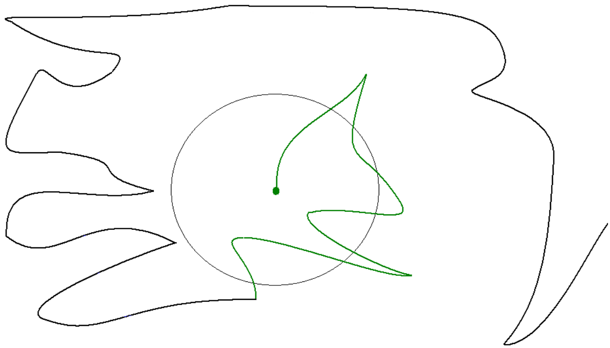

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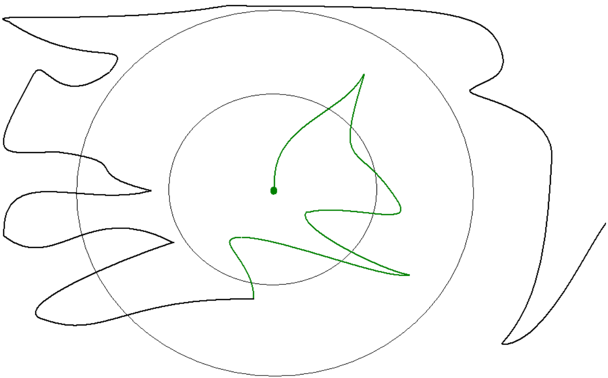

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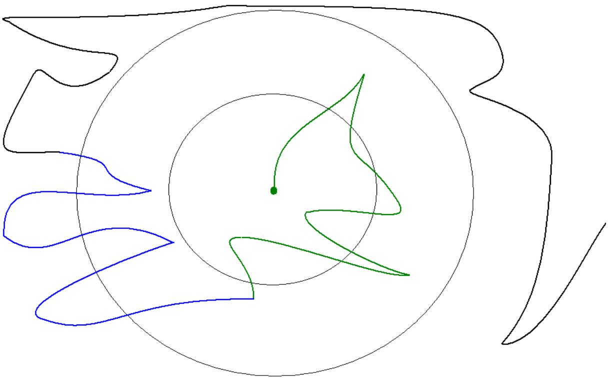

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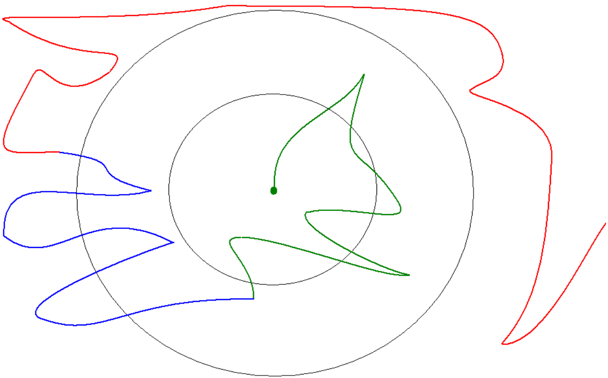

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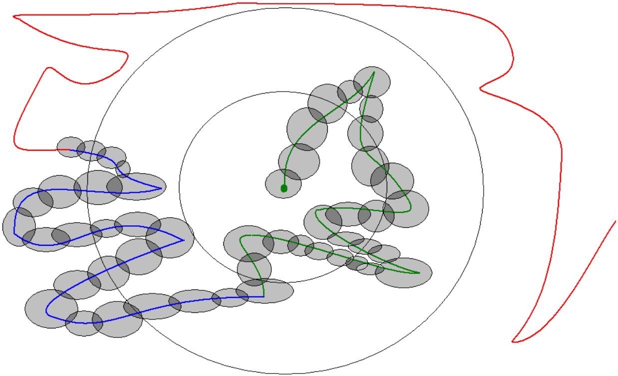

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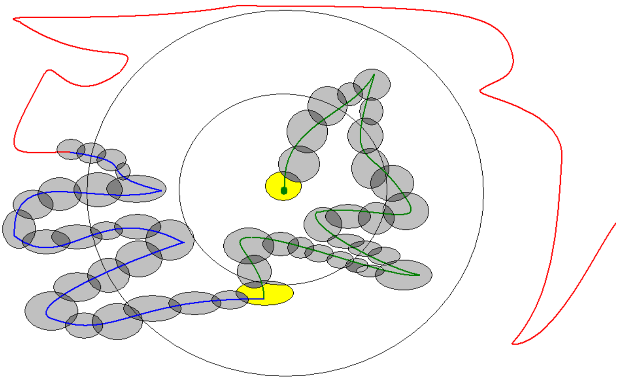
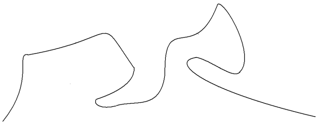


Topological lines

1. A **topological line** is a metric space homeomorphic to $\mathbb{R}$.



2. If L is a closed set homeomorphic to a topological line then "both of it's tails converge to infinity".

Problems

- A topological line is **not compact**!

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Computability of co-c.e. topological lines

Theorem

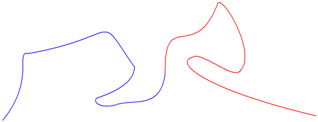
Let (X, d, α) be a nice computable metric space. Let L be a co-c.e. set such that L is a topological line. Then L is computable.

Proof(sketch)

Idea

Let L be a closed topological line. Let $f : \mathbb{R} \to L$ be a homeomorphism.

1. *For each $r \in \mathbb{R}$ the sets $f(\langle \infty, r])$ and $f([r, \infty\rangle)$ are topological rays.*



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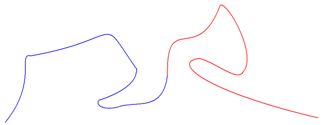

2. *If we find a computable $r \in \mathbb{R}$ such that $f(r)$ is computable and for which these sets are both co-c.e. we can apply the previous theorem.*

Proof(sketch)

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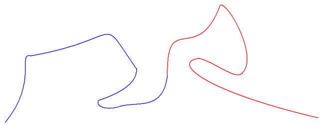

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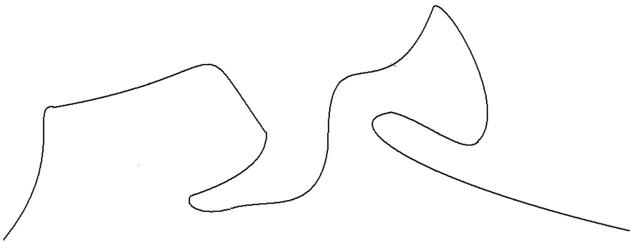
Problem

Such r might not exist!

Proof(sketch)



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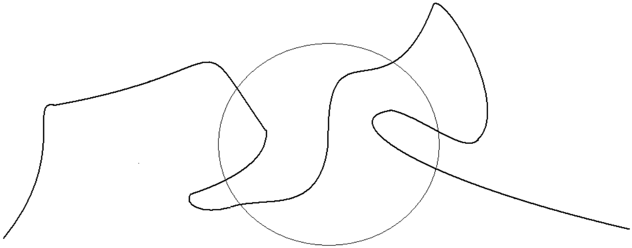

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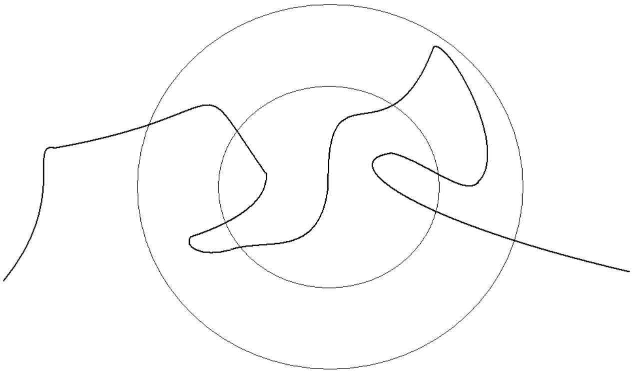

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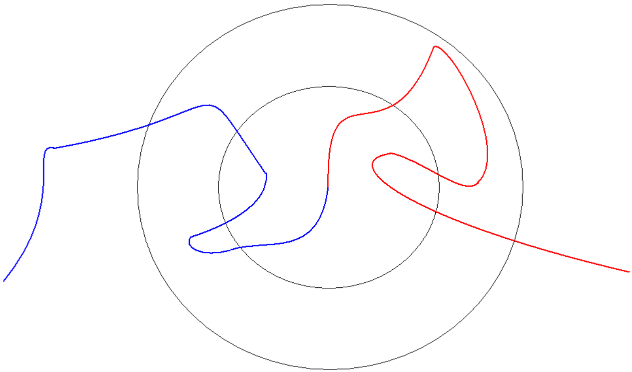

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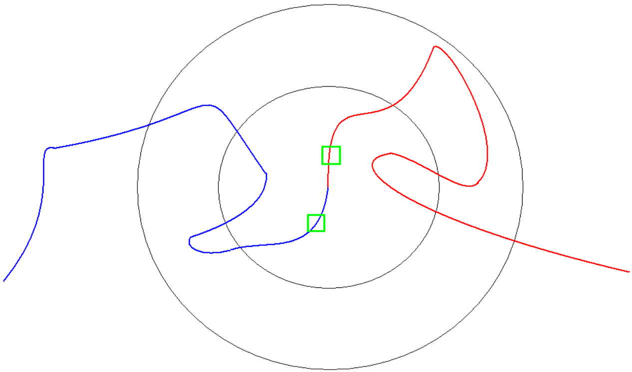

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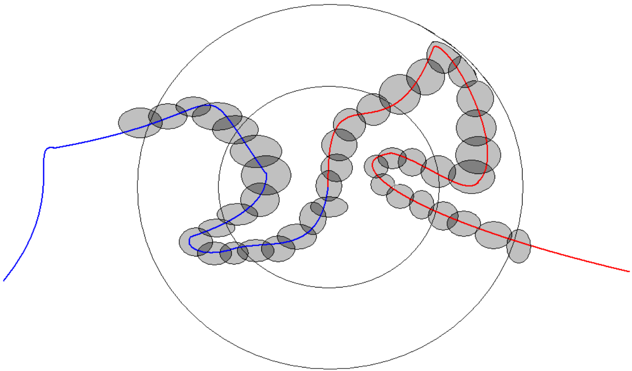

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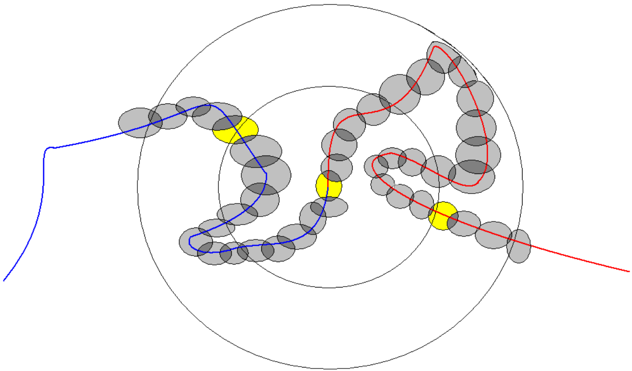
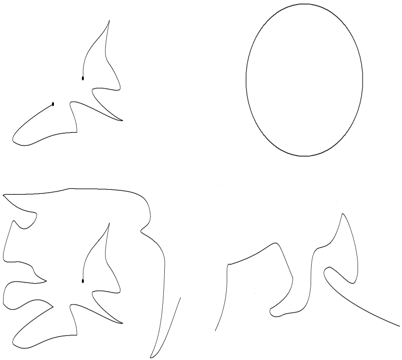


1-manifolds

Definition

- A **1-manifold with boundary** is a second countable Hausdorff topological space X in which each point has a neighborhood homeomorphic to $[0, \infty\rangle$.
- A **boundary** ∂X of X is the set of points $x \in X$ for which every homeomorphism between a neighbourhood of x and $[0, \infty\rangle$ maps x to 0.
- If $\partial X = \emptyset$ then X is a **1-manifold**.

1-manifolds



1-manifolds

- It is known that if X is a connected 1-manifold with boundary, then X is homeomorphic to $\mathbb{R}$, $[0, \infty\rangle$, $[0, 1]$ or the unit circle $\mathbb{S}^1$.

1-manifolds

Theorem

Let (X, d, α) be a nice computable metric space. Suppose M is a co-c.e. set which is a 1-manifold with boundary and such that M has finitely many components. Then the following implication holds:

$$\partial M \text{ computable} \implies M \text{ computable} .$$

In particular, each co-c.e. 1-mainfold in (X, d, α) with finitely many components is computable.

1-manifolds

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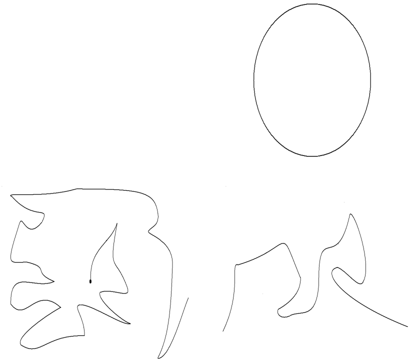
Remark

This theorem does not hold if we drop the assumtion that M has finitely many components!

Proof (sketch)



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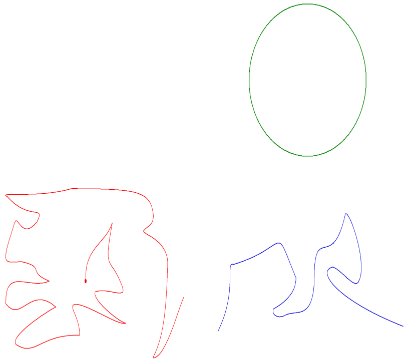

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Thank you!