Computational complexity of iterated maps on points and sets

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Discrete dynamical systems - point dynamics

- Let $M \subseteq \mathbb{R}^n$ be a cuboid, that is of the form $M = [a, b_1] \times \ldots \times [a_n, b_n]$ with $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R}$.
- Let $f : M \to M$ be a self mapping, in particular a $C^2$-diffeomorphism.
- Then the pair $(M, f)$ is called a discrete dynamical system.
- The dynamics is governed by the iteration equation

\[ x^{(k+1)} = f(x^{(k)}) \]

\[ x^{(0)} \in M. \]

- The second condition specifies the dynamics as an initial value problem.
- The initial value produces an orbit $(x^{(k)})_{k \in \mathbb{N}}$ under the dynamics.
Discrete dynamical systems - set dynamics

point dynamics - set dynamics

The standard definition as above describes a dynamics of \textit{points} in $\mathbb{R}^n$. However, dynamics can also be formulated for \textit{sets} of $\mathbb{R}^n$.

- Denote by $C_M$ the set of all compact subsets $A \subseteq M$ of $M$.
- Generalize $f: M \to M$ to $f_C: C_M \to C_M$ generically by setting $f_C(A) := f[A]$ for all $A \in C_M$.
- Then the pair $(C_M, f_C)$ defines a discrete dynamical system on sets.
- Note that if $f: M \to M$ is computable, then also $f_C: C_M \to C_M$ is computable.
Set dynamics - problems

Set dynamics in the case of **mixing**: Arnold’s cat map

- The map is *area preserving*.
- The initial set is uniformly spread over the whole domain in a few iterations.
- The number of spheres covering $A^{(k)}$ for given accuracy is typically growing *exponentially* in the number of iterations.
Consider a sphere $S = B(x^{(0)}, r)$ and examine $(Df^k)(x^{(0)})$. For given $k$ and $r$ sufficiently small, $f^k(x^{(0)}) + (Df^k)(x^{(0)})B(0, r) \approx f^k[S]$. 

\[ f^k(x^{(0)}) + (Df^k)(x^{(0)})(S-x^{(0)}) \approx f^k[S] \]
Local set dynamics by linearization: cocycles (2)

Lyapunov exponents

- The length of the axes of the ellipsoid $f^k(x^{(0)}) + (Df^k)(x^{(0)})B(0, r)$ are denoted by $r_1^{(k)}, \ldots, r_n^{(k)}$.
- The value $r_i^{(N)}$ measures the contraction or expansion near the orbit segment $(x^{(k)})_{k \leq N}$ along the $i$th principal axis.
- Then the $i$th Lyapunov exponent $\lambda_i$ is given by

$$\lambda_i = \lim_{k \to \infty} \frac{\ln(r_i^{(k)})}{k}$$

if the limit exists.
Local set dynamics by linearization: cocycles (3)

This linearized map can be expressed in the framework of a dynamical system by a cocycle:

\[ x^{(k+1)} = f(x^{(k)}) \]
\[ z^{(k+1)} = (Df)(x^{(k)}) \cdot z^{(k)} \]
\[ x^{(0)} \in \mathcal{M}, \quad z^{(0)} = \mathbb{1} \]

where \( \mathbb{1} \) is the \( n \times n \) identity matrix. Note that \( z^{(k)} = (Df^k)(x^{(0)}) \).
From approximations to enclosures

- The above linearization approximates $f^k[S]$.
- It is only asymptotically exact for $R \to 0$, but not for $R > 0$.
- However, this approach can be made rigorous even for $R > 0$.
- In verified computing, enclosures are used.
- Here, this is a function $\bar{f}_C : C_M \to C_M$ satisfying
  \[ f_C(A) \subseteq \bar{f}_C(A) \]
  for all $A \in C_M$.
- In the following it is convenient to restrict the domain of the enclosure to cuboids.
- Let $Q_M$ be the set of all cuboids $I \subseteq M$, then find an appropriate function $\bar{f}_Q : Q_M \to Q_M$ satisfying $f_C(A) \subseteq \bar{f}_Q(A)$ for all $A \in Q_M$. 
Finding an appropriate enclosure (1)

- Use a Taylor polynomial with remainder term:

\[ f_i(y) = f_i(x) + \sum_j \frac{\partial f_i}{\partial x_j} (x + \Theta_i(y - x))(y_j - x_j) \]

for \( x, y \in M \) with \( \Theta_i \in [0, 1] \), \( i = 1, \ldots, n \).

- Furthermore assume a **Lipschitz condition**:

\[ \left| \frac{\partial f_i}{\partial x_j}(y) - \frac{\partial f_i}{\partial x_j}(x) \right| \leq (L(I))_{ij} \cdot \| y - x \|_\infty \]

- Then

\[ f_C(I) \subseteq f(x) + ((Df)(x) + [-1, 1] \cdot |I| \cdot L(I))(I - x) \]

for all \( I \in Q_M \) where \( |I| = \sup_i(|I_i|) = \sup_i(b_i - a_i) \).
Finding an appropriate enclosure (2)

- Assume the following *normal form*

\[ I = x + [-1, 1] \cdot e \]

for cuboids \( I \in Q_M \), where \( x \in M \), \( e \in \mathbb{R}^n_+ \).

- Then

\[ f_C(I) \subseteq f(x) + [-1, 1] \cdot V(x, e) \cdot e \]

where

\[ V(x, e) = |(Df)(x)| + 2\|e\|_{\infty} L(x + [-1, 1] \cdot e). \]

- Reformulation: let

\[ CQ_M = \{(x, e) \in M \times \mathbb{R}^n_+ \mid x + [-1, 1] \cdot e \in Q_M \}, \]

then define \( \overline{f}_Q \colon CQ_M \to M \times \mathbb{R}^n_+ \) by

\[ \overline{f}_Q(x, e) = (f(x), V(x, e) \cdot e). \]
The modified cocycle

This set dynamics can be formulated by a *modified cocycle*:

\[
\begin{align*}
    x^{(k+1)} &= f(x^{(k)}) \\
    z^{(k+1)} &= V(x^{(k)}, e^{(k)}) \cdot z^{(k)} \\
    x^{(0)} &\in M, \quad z^{(0)} = 1
\end{align*}
\]

where

\[
\begin{align*}
    V(x, e) &= |(Df)(x)| + 2\|e\|_{\infty} L(x, e) \\
    e^{(k)} &= z^{(k)} \cdot e^{(0)} \\
    e^{(0)} &\in \mathbb{R}^n \text{ s.t. } (x^{(0)}, e^{(0)}) \in CQ_M.
\end{align*}
\]
The model of computation - representing reals

- We start with a **fixed point number system**:

\[
\hat{\mathbb{R}}(p, \beta) = \{ x \in \mathbb{R} \mid \exists r, s \in \mathbb{Z} . x = s + r \cdot \beta^{-p} \land |r| \leq \beta^{p} - 1 \} 
\]

- where \( \beta \geq 1 \) is the **base** and \( p \geq 1 \) the **precision**.

- Then we allow **arbitrary precision**:

\[
\hat{\mathbb{R}}(\beta) = \bigcup_{p \geq 1} \hat{\mathbb{R}}(p, \beta)
\]

- A fixed point number \( \hat{x} \in \hat{\mathbb{R}}(p, \beta) \) **approximates** a real \( x \in \mathbb{R} \), if

\[
x \in \hat{x} + \beta^{-p}[-1, 1].
\]

- Any \( x \in \mathbb{R} \) can be **represented** by a sequence \( (\hat{x}_n)_{n \in \mathbb{N}} \) with \( \hat{x}_n \in \hat{\mathbb{R}}(p_n, \beta) \), each \( \hat{x}_n \) approximating \( x \) and \( \lim_{n \to \infty} p_n = \infty \).
The model of computation - representing functions

- Let \( f : \subseteq \mathbb{R}^n \to \mathbb{R} \) be given. A function \( \hat{f} : \subseteq \hat{\mathbb{R}}^n \to \hat{\mathbb{R}} \) is called an approximation function for \( f \) if:

\[
\hat{x} \in \hat{\mathbb{R}}^n \text{ approximates } x \in \text{dom}(f) \implies \hat{f}(\hat{x}) \text{ approximates } f(x).
\]

- We call \( \hat{f} \) approximation continuous if for any \((\hat{x}_n)_{n\in\mathbb{N}}:\)

\[
(\hat{x}_n)_{n\in\mathbb{N}} \text{ representing } x \in \text{dom}(f) \implies (\hat{f}(\hat{x}_n))_{n\in\mathbb{N}} \text{ representing } f(x)
\]

- Since \( \hat{\mathbb{R}} \) is countable, define computability for \( \hat{f} : \subseteq \hat{\mathbb{R}}^n \to \hat{\mathbb{R}} \) by classical computability theory.

- Then \( f \) is called computable, if it has a computable approximation continuous approximation function.
Further specifications and generalizations

- Generalization of an approximation of $x \in \mathbb{R}$:
  \[(\hat{x}, e) \in \hat{\mathbb{R}}(p_1, \beta) \times \hat{\mathbb{R}}(p_0, \beta) \text{ approx. } x \in \mathbb{R} \iff x \in \hat{x} + [-1, 1] \cdot \bar{e}\]

- **Normal form** for approximation functions $\hat{f} : \subseteq \hat{\mathbb{R}}^n \rightarrow \hat{\mathbb{R}}$:
  \[\hat{x} \in \hat{\mathbb{R}}(p_1, \beta) \times \ldots \times \hat{\mathbb{R}}(p_n, \beta) \implies \hat{f}(\hat{x}) \in \hat{\mathbb{R}}(p_0, \beta)\]
  \[p_0 = \min(p_1, \ldots, p_n)\]

- Normal form for approximations of **self mappings** $f : M \rightarrow M$:
  \[\hat{x} \in \hat{\mathbb{R}}(p_1, \beta) \times \ldots \times \hat{\mathbb{R}}(p_n, \beta) \implies \hat{f}(\hat{x}) \in \hat{\mathbb{R}}(p'_1, \beta) \times \ldots \times \hat{\mathbb{R}}(p'_n, \beta)\]
  \[\beta^{-p'} \leq |Q \cdot \beta^{-p}|\]

  where $Q \in \mathbb{R}^{n \times n}$ is an **orthogonal matrix**.
Point dynamics - formulating the algorithm (1)

- A **finite segment** \((x^{(k)})_{0 \leq k \leq N}\) of length \(N\) of the **true orbit** is computed: a **pseudo orbit** \((\hat{x}^{(k)})_{0 \leq k \leq N}\)
- with demanded precision \(p_1^o, \ldots, p_n^o\): \(x_i^{(k)} \in \hat{x}_i^{(k)} + \beta^{-p_i^o}[-1, 1]\) for **all** \(k \leq N\).
- In the above formulation:
  \[
  x^{(k)} \in \hat{x}^{(k)} + [-1, 1] \cdot \overline{e}^{(k)}, \quad \overline{e}^{(k)} \leq \beta^{-p^o}.
  \]
- Since \(N\) is fixed, there ex. \(p_1^s, \ldots, p_n^s \geq 1\) s.t. the above condition is fulfilled starting with
  \[
  \hat{x}^{(0)} \in \hat{\mathbb{R}}(p_1^s, \beta) \times \ldots \times \hat{\mathbb{R}}(p_n^s, \beta)
  \]
  and approximating \(f\) by
  \[
  \hat{x}^{(k)} \in \hat{\mathbb{R}}(p^{(k)}, \beta) \implies \hat{f}(\hat{x}) \in \hat{\mathbb{R}}(p^{(k+1)}, \beta)
  \]
  \[
  \beta^{-p^{(k+1)}} \leq \overline{e}^{(k+1)} \leq |Q^{(k+1)}\beta^{-p^s}|.
  \]
Point dynamics - formulating the algorithm (2)

- Using the above precision control for \( \hat{f} \) and the modified cocycle for calculating the error propagation leads to

\[
\bar{e}^{(k+1)} \geq V(\hat{x}^{(k)}, \bar{e}^{(k)})\bar{e}^{(k)} + |Q^{(k+1)}\beta^{-p_s}|
\]

for the recursion of the error.

- Finally, by approximating \( Df \) by \( \hat{D}f \),

\[
\bar{e}^{(k+1)} = \overline{V}(\hat{x}^{(k)}, \bar{e}^{(k)})\bar{e}^{(k)} + |Q^{(k+1)}\beta^{-p_s}|
\]

is obtained where

\[
\overline{V}(\hat{x}, \bar{e}) = |(\hat{D}f)(\hat{x})| + \|\bar{e}\|_\infty (2 \cdot \overline{L}(\hat{x}, \bar{e}) + E).
\]
Set dynamics - differences in the algorithm

- By pairing \((x^{(k)}, z^{(k)})\), the modified cocycle can be viewed as a new dynamical system: then set dynamics is reduced to point dynamics with different phase space.

- Alternatively, the pair \((\hat{x}^{(k)}, e^{(k)})\) is interpreted as an enclosure for cuboids:

\[
l^{(k)} = x^{(k)} + [-1, 1] \cdot e^{(k)} \subseteq \hat{x}^{(k)} + [-1, 1] \cdot \overline{e}^{(k)}.
\]

- Then the error control has to be reinterpreted.

- But the resulting formulas are nearly the same as in the case of points.

- Only the interpretation is different.
Point dynamics - set dynamics (enclosures)

\[
\hat{x}^{(k+1)} = \hat{f}(\hat{x}^{(k)})
\]

\[
\overline{z}^{(k+1)} = \overline{V}(\hat{x}^{(k)}, \overline{e}^{(k)}) \overline{z}^{(k)} + |Q^{(k+1)}|
\]

\[
\hat{x}^{(0)} \in \hat{\mathbb{R}}(p_{i}^{s}, \beta) \times \ldots \times \hat{\mathbb{R}}(p_{n}^{s}, \beta)
\]

\[
\overline{V}(\hat{x}, \overline{e}) = |(\hat{D}f)(\hat{x})| + \|\overline{e}\|_{\infty} (2 \cdot \overline{L}(\hat{x}, \overline{e}) + E)
\]

\[
\overline{e}^{(k)} = \overline{z}^{(k)} \overline{e}^{(0)}
\]

\[
\overline{e}^{(0)} = \alpha \cdot \beta^{-p^{s}}, \quad \overline{z}^{(0)} = 1
\]

- Even the algorithm can be expressed via a modified cocycle.
- $\alpha = 1$ in the case of points, $\alpha = 2$ in the case of cuboids.
- $p^{s}$ is the initial precision for $\hat{x}^{(0)}$ in the case of points.
- In the case of cuboids, $p^{s}$ determines an upper bound on the extent of $I^{(0)}$.
Computational complexity: loss of significance rates

Definition

Let $p_i^{\text{min}}(x, N, p^o)$ be the minimal precisions for $p_i^s$ such that the demanded precision $p^o$ for the pseudo orbit of length $N$ is achieved when $x \in M$ is the initial condition.

- The growth rate of $p_i^{\text{min}}(x, N, p^o)$ is

$$
\sigma(x, p^o) = \limsup_{N \to \infty} \frac{p_i^{\text{min}}(x, N, p^o)}{N}.
$$

The loss of significance rates $\sigma: M \to \mathbb{R}^n$ are defined by

$$
\sigma(x) = \lim_{p \to \infty} \sigma(x, p). \quad (1)
$$
Loss of significance rate - main results (1)

An easy provable observation: the loss of significance rates are **bounded**. A bit more effort: the loss of significance rates are bounded from below by the **Lyapunov exponents**.

**Proposition**

Let \((M, f)\) be a dynamical system, \(x \in M\) and \(\sigma(x)\) the loss of significance rates. Then there exist some \(c \in \mathbb{R}_+^n\) such that \((0, \ldots, 0)^t \leq \sigma(x, p) \leq \sigma(x) \leq c\) holds for all precisions \(p_1, \ldots, p_n \geq 1\).

**Theorem**

Let the notation as above. Then

\[
\sigma_i(x) \geq \frac{1}{\ln(2)} \lambda_i(x)
\]

holds for \(i = 1, \ldots, n\) where \(\lambda_i(x)\) is the \(i\)th Lyapunov exponent, if it exists.
The proof of the theorem is based on a $QR$-decomposition of the form

$$Q^{(k+1)}R^{(k+1)} = (Df)(x^{(k)})Q^{(k)}$$

$$Q^{(0)} = I.$$

The link between this $QR$-decomposition and the Lyapunov exponent is well established in the literature.

Also $R^{(k)}$ and $Q^{(k)}$ are approximated by $\hat{R}^{(k)}$, $\hat{Q}^{(k)}$.

Since in $\hat{e}^{(k)}$ not $(Df)(.)$ but $|\hat{(Df)}(.)|$ is relevant the matrix multiplication of matrices of the form $|A|$ need to be done more elaborate to reduce overestimation of the error.
Loss of significance rate - main results (3)

If some additional aspects concerning the $QR$-decomposition and the Lyapunov exponents turn out to be true (which actually have not been checked yet), the we also have:

**Theorem**

*Let the notation as above. Then*

\[
\sigma_i(x) \leq \frac{1}{\ln(2)} \lambda_i(x)
\]

*holds for $i = 1, \ldots, n$.*

Thus the Lyapunov exponents turn out also to be an upper bound on the loss of significance rates.