# Sixteenth International Conference on Computability and Complexity in Analysis

## UNIVERSITY OF ZAGREB

July 8-11, 2019
Zagreb, Croatia

## INVITED SPEAKERS

**Hannes Diener** (Christchurch, New Zealand)
**Jacques Duparc** (Lausanne, Switzerland)
**Hugo Férée** (Kent, UK)
**Guido Gherardi** (Bologna, Italy)
**Bruce Kapron** (Victoria, Canada)
**Joël Ouaknine** (Saarbrücken, Germany)
**Svetlana Selivanova** (Daejeon, Republic of Korea and Novosibirsk, Russia)

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# Programme

**Monday, July 8**

8:30am Registration
9:00am Opening
9:30am Invited talk - *Hannes Diener*

10:30am Coffee break

11:00am *Russell Miller*: A computability-theoretic proof of Lusin's Theorem
11:30am *Matthew de Brecht, Arno Pauly and Matthias Schröder*: Overt choice
12:00pm *Lars Kristiansen*: On Subrecursive Representability of Irrational Numbers: Continued Fractions and Contraction maps

12:30pm Lunch

2:30pm Invited talk - *Jacques Duparc*

3:30pm Coffee break

4:30pm *Peter Hertling and Philip Janicki*: A weakly computable number which can be only decomposed into random left-computable numbers
5:00pm *Hyunwoo Lee, Sewon Park and Martin Ziegler*: Is Brownian Motion Computable?

**Tuesday, July 9**

9:30am Invited talk - *Hugo Férée*

10:30am Coffee break

11:00am *Ivan Georgiev*: On representations of irrational numbers in subrecursive context
11:30am *Matthias Schroeder*: On maximal Co-Polish Spaces
12:00pm *Arno Pauly*: Effective local compactness and the hyperspace of located sets

12:30pm Lunch

2:30pm Invited talk - *Guido Gherardi*
3:30pm *Florian Steinberg and Holger Thies*: Formal proofs about metric spaces and continuity in coqrep

4:00pm Coffee break

4:30pm *Konrad Burnik and Zvonko Iljazović*: Density of maximal computability structures
5:00pm *Zvonko Iljazović and Matea Jelić*: Computability of spaces with attached arcs

**Wednesday, July 10**

9:30am Invited talk - *Joël Ouaknine*

10:30am Coffee break

11:00am *Alonso Herrera*: Computability of topological entropy in the logistic family
11:30pm *Robert Rettinger and Xizhong Zheng*: Convergence Dominated Reducibility and Divergence Bounded Computable Real numbers

12:00pm Lunch

2:00pm Excursion

7:00pm Conference dinner

**Thursday, July 11**

10:00am Invited talk - *Bruce Kapron*

11:00am Coffee break

11:30am Invited talk - *Svetlana Selivanova*
12:30am Closing

12:45pm Lunch

# Invited Talks

# Completeness is Overrated (. . . Sometimes)

Hannes Diener*
joint work with Matthew Hendtlass*

June 26, 2019

It is a common theme in computable/constructive analysis that theorems which are not (uniformly) computable often can be made so by adding the assumption that the underlying space is complete. In Bishop style constructive mathematics this is cryptically known as the "lambda technique", which owes its name to the common pattern of choosing a binary sequence—traditionally labelled lambda—which in turn is used to construct a Cauchy sequence.

More specific: the crucial construction involves a sequence $(x_n)_{n\geqslant 1}$ converging to a limit $x$ and a binary, increasing sequence $(\lambda_n)_{n\geqslant 1}$ which switches to 1 if—vaguely speaking—something interesting happens. We then combine all of these into a new sequence $((\lambda \circledast x)_n)_{n\geqslant 1}$ by setting

$$(\lambda \circledast x)_n = \begin{cases} x_m & \text{if } \lambda_n = 1 \text{ and } \lambda_m = 1 - \lambda_{m+1} \\ x_\infty & \text{if } \lambda_n \end{cases}$$

Any such constructed sequence is automatically Cauchy, so if we are working in a complete space it converges. In this talk we are going to show that in most cases a weakened form of completeness is actually sufficient. We will show that there is a plethora of examples to which this generalisation applies. The most prominent examples will be the Kreisel-Lacome-Shoenfield theorem [4], which states that all (computable) real-valued functions on a complete metric space are continuous, and the problem of basic path glueing, which fundamentally underlies the development of homotopy theory.

Of course, none of this would be interesting, if there were no good examples of spaces that are computably complete in the weakened sense but not complete. Indeed, we will show that there are many natural spaces falling into this category.

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# Continuous Reductions on the Cantor Space and on the Scott Domain

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### Extended abstract

The Cantor space – $2^{\mathbb{N}}$ – and the Scott domain – $\mathcal{P}(\omega)$ – are two topological spaces whose points are sets of integers. But if the Cantor space deals with the set of subsets of integers while equipped with a topology of positive and negative information (conveyed through their characteristic functions via the product topology of the discrete topology on $\{0,1\}$), the Scott domain drops that condition of negative information only to keep the one of positive information through the topology generated by the basis $\{\mathcal{O}_F \mid F \subseteq \mathbb{N},\ F \text{ finite}\}$ where $\mathcal{O}_F = \{A \subseteq \mathbb{N} \mid F \subseteq A\}$.

As a consequence, the Scott domain is not anymore Hausdorff ($T_2$), not even Fréchet ($T_1$) but only Kolmogorov ($T_0$). So, on one hand it seems far away from the Cantor space which is a complete separable metric space (i.e., a Polish space) for the reason it is not even metrizable. But on the other hand, the Scott domain is a complete separable quasi-metrizable[1] space (i.e., a quasi-Polish space). Moreover, if the Cantor space is universal for 0-dimensional Polish space[2], the Scott domain is universal for all quasi-Polish space[3] as shown by de Brecht [2]. For instance, the Cantor space is homeo-

[1]A quasi- metric is a metric without the symmetry condition: $d(x,y) = d(y,x)$.

[2]Every 0-dimensional Polish space is homeomorphic to some $\mathbf{\Pi}^0_2$ subset of $2^{\mathbb{N}}$.

[3]Every quasi-Polish space is homeomorphic to some $\mathbf{\Pi}^0_2$ subset of $\mathcal{P}(\omega)$.

morphic to the following set

$$\{A \subseteq \mathbb{N} \mid \forall n \in \mathbb{N} \ \ (2n \in A \iff 2n+1 \notin A)\}.$$

More results by de Brecht suggest that a reasonable descriptive set theory still holds in the quasi-Polish setting. However, very little is known about the Wadge order in this context [5, 6, 1, 4]. The Wadge order $\leq_W$ — named after Bill Wadge [7] — is the quasi-order induced by reductions via continuous functions that compares the topological complexity of the subsets of a topological space $X$ . i.e., given $A, B \subseteq X$, $A \leq B$ holds if there exists a continuous function $f : X \to X$ such that for all $x \in X$, $x \in A \iff f(x) \in B$.

We outline the main features of the Wadge order on the Cantor space – the beautiful Wadge hierarchy – and on the Scott domain – not even a well quasi order (a result due to Louis vuilleumier [3]) – and compare these two.

# References

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# A Higher-Order Approach of Complexity in Computable Analysis

Hugo Férée

**Second-Order Complexity in Analysis** Mainly focused on finite objects, the notion of computability has been generalised to most domains of interest mostly via the Type-two Theory of effectivity [Wei00] which relies on second-order computations.

Complexity, however, has only been extended to computable analysis on a case-by-case basis, starting for example with real numbers and real functions [Ko91] or more recently with real operators [KC10].

Let us recall that a general definition for complexity could be: *a bound on computation time* (for example) *with respect to a bound on the size of the input.* The main difficulty here is then to define a relevant notion of size for the inputs. Kapron and Cook [KC96] provide such a definition for first-order functions, which together with the oracle Turing machine model induce a relevant notion of complexity for second-order computations.

It is then relatively straightforward to generalise this to define the complexity of a function between two represented spaces as the (second-order) complexity of its realiser. But this approach does not always induce a relevant notion of complexity.

**Limitations of Second-order Computations** Indeed, one could expect that given a represented function space, its application function is computable in feasible time. But we have shown [FH13] that the running time of the application function of certain such spaces cannot even be bounded with respect to the size of its inputs, whichever representation function this space is equipped with.

Intuitively, such spaces are intrinsically second-order spaces, like finite sets are of order 0 as well as real numbers and real functions are of order 1 (they can be meaningfully represented by first-order functions). This suggests a potential solution: using higer-order representation spaces instead of first-order ones ($\Sigma^{*\Sigma^*}$ for example).

**Higher-Order Complexity Theory** The main obstacle to this idea is that there was no generic notion of complexity for higher-order functions until recently. Indeed, there are already several models of computation for such objects, but the main missing ingredient was, once again, a relevant notion of size.

We have thus proposed [Fér17] such a definition using game semantics [Nic+96; HO00]. This framework can be seen as a way of representing a computation as a dialogue between a machine and its input, where each of them can ask finite amounts of information about the other, in similar way to an oracle Turing machine which can query its input at given points. We obtain a general definition of complexity for higher-order functions (namely PCF), as well as a class of polynomial-time computable functions, which satisfies the basic properties that one should expect from it.

Our goal will now be to study this new, promising definition and see whether we can use it in a "*Higher-type Theory of Effectivity*", extending TTE to handle complexity in a meaningful way.

## References


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# Total Weihrauch reducibility

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In computable analysis usually problems with partial domains are investigated. Such problems are defined by formal statements of the form

$$(\forall x \in X)(x \in D \Rightarrow (\exists y \in Y)P(x,y))$$

which show that the existence of a $y$ such that $P(x,y)$ depends on the satisfaction of the premise $x \in D$.

There are nevertheless natural ways to make problems independent of the their premises. A simple idea is to accept every $y \in Y$ as a valid output when the premise $x \in D$ is not satisfied. This is in agreement with the *ex falso quodlibet* logical principle according to which the choice of $y$ is irrelevant for the truth of the conditional $x \in D \Rightarrow P(x,y)$ when $x \notin D$. This automatically suggests a possible candidate for a modification of our original problem meeting our requirement:

**Definition 0.1 (Totalization)** *For every problem $f :\subseteq X \rightrightarrows Y$ the* totalization *of $f$ is defined as*

$$\mathsf{T}f : X \rightrightarrows Y, x \mapsto \begin{cases} f(x) & \text{if } x \in \mathrm{dom}(f) \\ Y & \text{otherwise.} \end{cases}$$

An interesting example is given by the totalizations $\mathsf{TC}_X$ of the closed choice operators $\mathsf{C}_X :\subseteq \mathcal{A}_-(X) \rightrightarrows X, A \mapsto A$, accepting any $x \in X$ as a valid solution for the input $\emptyset$.

Since problems usually have partial domains (as well as space representations), realizers are assumed to be in general partial functions; in other words, for a realizer $F$ of $f$ the existence of $F(p)$ is not guaranteed if $p$ does not denote an element in $\mathrm{dom}(f)$. On the contrary, in this talk I will use total realizers to introduce the notion of *total Weihrauch reducibility*:

**Definition 0.2 (Total Weihrauch reducibility)** *Let $f :\subseteq X \rightrightarrows Y$ and $g :\subseteq U \rightrightarrows V$ be problems. We define:*

$$f \leq_{\mathrm{tW}} g :\Longleftrightarrow (\exists \text{ computable } H, K :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}})(\forall G \vdash_{\mathrm{t}} g)\ H\langle \mathrm{id}, GK\rangle \vdash_{\mathrm{t}} f.$$

(here $\vdash_{\mathrm{t}}$ means that realizers are assumed to be *total*; the *strong* total Weihrauch reducibility $\leq_{\mathrm{stW}}$ is defined analogously). In this definition, the representations of the spaces $X, Y, U, V$ are replaced by computably equivalent precomplete representations. Intuitively, a precomplete representation of a space allows us to see all computable realizers of problems with range in that space as total:

**Definition 0.3 (Precompleteness)** $\delta :\subseteq \mathbb{N}^{\mathbb{N}} \to X$ is said to be a *precomplete representation*, if for any computable $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ there exists a total computable $G : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that

$$\delta F(p) = \delta G(p)$$

for all $p \in \mathrm{dom}(F)$.

Total Weihrauch reducibility can be characterized in terms of the usual Weihrauch reducibility by using the completion functional $f \mapsto \overline{f}$ defined through space completions. The *completion* $\overline{X}$ of a represented space $X$ is given by the space $X \cup \{\bot\}$, for $\bot \notin X$, equipped with a fixed precomplete representation. Space completions have been studied by D. Dzhafarov in [1]. The completion of a problem is then defined as follows:

**Definition 0.4 (Completion)** *Let* $f :\subseteq X \rightrightarrows Y$ *be a problem. We define the* completion *of* $f$ *by*

$$\overline{f} : \overline{X} \rightrightarrows \overline{Y}, x \mapsto \begin{cases} f(x) & \text{if } x \in \mathrm{dom}(f) \\ \overline{Y} & \text{otherwise.} \end{cases}$$

It holds:

**Lemma 0.5 (Completion and total Weihrauch reducibility)** $f \leq_{\mathrm{tW}} g \iff f \leq_{\mathrm{W}} \overline{g}$ *for all* $f$ *and* $g$ *(and analogously for* $\leq_{\mathrm{stW}}$*).*

It can be shown that the completion operator is a closure operator. Although $f \equiv_{\mathrm{stW}} \overline{f}$ holds for every problem $f$, a natural question to ask is, for a given $f$, whether $f \equiv_{\mathrm{W}} \overline{f}$ (we call the problems satisfying this equivalence *complete*). Complete problems are important, since they determine the same cone both with respect to ordinary Weihrauch reducibility and total reducibility. In this talk I will show examples of problems that are complete, and others that are not, with a particular emphasis on the choice principles $\mathsf{C}_X$. Strictly related, I will analyze which computational classes (non deterministic, Las Vegas, with finitely many mind changes,...) are preserved downwards by total Weihrauch reducibility.

Similarly, I will investigate which choice operators $\mathsf{C}_X$ are Weihrauch equivalent to their totalizations $\mathsf{TC}_X$; moreover, operators $\overline{\mathsf{C}_X}$ and $\mathsf{TC}_X$ will be compared with each other.

Dual to the notion of completeness is that of co-completeness:

**Definition 0.6 (Co-completeness)** $f$ *is called* co-complete *if* $f \leq_{\mathrm{W}} \overline{g} \iff f \leq_{\mathrm{W}} g$ *for all* $g$*.*

Likewise one can define the notion of *co-totality* and the corresponding strong versions.

The relationships between completeness and co-completeness will be discussed and important examples of co-complete problems will be presented.

Total Weihrauch reducibility and problem completions are far from being merely arbitrary notions with no particular interesting application. They allow us to transform the ordinary Weihrauch lattice into a Brouwer algebra, and even more, into a model of Jankov logic (intuitionist logic plus the law of the weak excluded middle $\neg A \vee \neg\neg A$). To obtain that, one needs to define suitable conjunction, disjunction and implication operators over the structure of the *parallelized total Weihrauch degrees* induced by the reduction relation $\leq_{\mathrm{ptW}}$ (where $f \leq_{\mathrm{ptW}} g \iff f \leq_{\mathrm{tW}} \widehat{g}$). This is interesting, since Higuchi and Pauly [2] had previously proved that the (ordinary) parallelized Weihrauch lattice is not a Brouwer algebra.

(*Joint work with Vasco Brattka*)

## References


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# Type-two Feasibility via Bounded Query Revision

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The problem of generalizing the notion of polynomial time to a type-two setting, where inputs include not only finitary data, e.g., numbers or strings of symbols, but also functions on such data, was first posed by Constable in 1973 [3], where he also gave a possible characterization of such a class. Subsequently, Mehlhorn [13], using a generalization of Cobham's scheme-based approach [2], gave a characterization of a class $\mathcal{L}()$ which satisfied a *Ritchie-Cobham property*, showing robustness with respect to oracle Turing machine (OTM) computation. It was later shown by Clote [1] that Mehlhorn's class was a proper extension of Constable's. While the Ritchie-Cobham property provided some computational intuition for Mehlhorn's class, the question of generalizing familiar the type-one notion of poly-time TM computation remained open. Almost two decades later, Kapron and Cook [7] were finally able to give such a characterization, using the notions of *function length* and *second-order polynomials*. In particular, for a function $\varphi : \Sigma^* \to \Sigma^*$, they defined the function $|\varphi| : \omega \to \omega$ by $|\varphi|(n) = \max_{|\mathbf{a}| \le n} |\varphi(\mathbf{a})|$. They proved that the functionals computable in polynomial time with respect to the length of function and string inputs coincide exactly with $\mathcal{L}()$.

The characterization provided by [7] still had some shortcomings. The first is that the type-two functional $\lambda\varphi\lambda\mathbf{a}.|\varphi|(|\mathbf{a}|)$ is itself not in $\mathcal{L}()$. Reasoning about resource bounds involving function lengths and second-order polynomials can be difficult, and unintuitive. A second criticism is that the notion of function length may be viewed as being somewhat *ad hoc*, and may not be adequate if the goal is to provide a general account of *feasibility* for type-two functionals. With respect to this criticism, Cook in [4] proposes a notion of *intuitive feasibility* based on two conditions: (1) *Oracle polynomial time (OPT)* computability and (2) preservation of type-one polynomial time. He also demonstrated the existence of a *well-quasiordering functional* $L$ that satifies these conditions but is not in $\mathcal{L}()$. Subsequent work cast doubt on whether these conditions are restrictive enough. In particular Seth [14] showed that $L$ does not preserve the Kalmar elementary functions, and that more powerful forms of (2), in combination with the Ritchie-Cobham property, lead back to Mehlhorn's class $\mathcal{L}()$.

Beyond its application in the definition of inuitive feasibility, OPT is interesting in its own right, as a minimal requirement for type-two feasibility. It is also the starting point for the results we present here. It is important to note that OPT alone is too lax a notion to characterize type-two feasibility: in particular, the functional $\lambda\varphi\lambda\mathbf{a}\lambda\mathbf{c}.\varphi^{|\mathbf{c}|}(\mathbf{a})$ is in OPT, but when applied to the function argument $\lambda\mathbf{a}.\mathbf{aa}$ (self-concatenation) results in a function with exponential growth. Motivated in part by the well-known difficulties of working with second-order polynomials, Kawamura and Steinberg [11] considered a restriction version of OPT which was adequate for some applications in feasible analysis (e.g., complexity of operators as studied in [10].) From the above example, an obvious problem with OPT is that is allows unbounded oracle growth, that is, the answer returned by the oracle may increase in size with every call and result in a computation which still satisfies a OPT run-time bound. A natural mitigation is to limit by a constant the number of times an answer of increasing size may be returned. The resulting class was dubbed *strong polynomial time (SPT)*, and its utility and equivalence to full $\mathcal{L}()$ for *length-monotone* function inputs was demonstrated.

Unfortunately, SPT does not capture $\mathcal{L}()$ for arbitrary function inputs. A simple example is the functional $\lambda\varphi\lambda\mathbf{a}.\max_{\mathbf{b}\subseteq\mathbf{a}}\varphi(\mathbf{b})$, where $\subseteq$ denotes string prefix. Clearly, for any query strategy there is an input $\varphi$ which violates the SPT condition. On the other hand, this functional is computable in OPT in such a way that *queries* to the function input do not increase in size more than a constant number of times (in fact they may be made in stricty decreasing $\subseteq$-order.) Starting from this observation Kapron and Steinberg [9] proposed a new class dubbed *moderate polynomial time (MPT)* analagously to SPT but with respect to query size. MPT is also strictly contained in $\mathcal{L}()$, although the example in this case is more artificial. In both the case of SPT and MPT, it appears that their weakness is closely related to the fact that the classes are not closed under functional substitution (i.e. operator composition.) The main result of [9] shows that this is exactly the case. If for a class X of functionals containing all type-one poly-time functions we write $\lambda(\mathrm{X})_2$ for the 2-section of the $\lambda$-closure of X, then $\lambda(\mathrm{SPT})_2 = \lambda(\mathrm{MPT})_2 = \mathcal{L}()$.

In the above equivalence, the adequacy of SPT and MPT is proved by showing that the *Cook-Urquhart recursor* $\mathcal{R}$ [5] is contained in $\lambda(\mathrm{SPT})_2$ and in $\lambda(\mathrm{MPT})_2$. This recursor captures Cobham's *limited recursion on notation* at type level two. One well-studied drawback of such recursion schemes is the need for external bounding at each application of the step function. Here the approach of intrinsically bounding by a constant the number of size increases in the step function appears as an alternative. In [8], Kapron and Steinberg show that such an approach works. In particular, they define the iterator $\mathcal{I}^!_k(\varphi, \mathbf{a}, \mathbf{c}) = \varphi^\ell(\mathbf{a})$ where $\ell \leq |\mathbf{a}|$ is minimal such that the sequence of applications of $\varphi$ contains no more than $k$ size increases, and show that for every $k \geq 1$, $\lambda(\{\mathcal{I}^!_k\})_2 = \mathcal{L}()$. A similar result is shown for an iterator with bounded input revision.

Further evidence of the utility of bounded query revision is provided by [6]. This work builds on Marion's characterization of type-one polynomial time using an imperative langauge with a type system based on secure flow information analysis [12], and gives a tier-based typing system for an extension of Marion's language with oracles. The terminating programs that are typable in this language are in MPT, and the 2-section of the lambda closure of the functionals they define are exactly $\mathcal{L}()$. Type inference for this langauge is decidable in polynomial time, thus providing a framework for tractable reasoning about type-two feasibility.

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## On the Zeros of Exponential Polynomials

Joël Ouaknine, Max Planck Institute for Software Systems, Saarbrücken, Germany

Exponential polynomials are central objects of study in analysis, notably because they are solutions of linear differential equations with constant coefficients. In this talk, I examine some fundamental decision problems for real-valued exponential polynomials, such as the existence of finitely many zeros, infinitely many zeros, and divergence at infinity. Although the decidability and complexity of such problems are open, some partial and conditional results are known, occasionally resting on certain number-theoretic hypotheses such as Schanuel's conjecture. More generally, the study of algorithmic problems for exponential polynomials (or equivalently regarding the behaviour of linear dynamical systems) draws from an eclectic array of mathematical tools, ranging from Diophantine approximation to algebraic geometry. I will present a personal overview of the field and discuss areas of current active research.

This is joint work with James Worrell, Ventsi Chonev, and Joao Sousa-Pinto.

# Computational Complexity of PDEs

Svetlana Selivanova

Supported by Korean research foundation grant NRF-2017R1E1A1A03071032 and by the Korean Ministry of Science and ICT grant NRF-2016K1A3A7A03950702.

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We investigate complexity bounds for computing solutions to initial-value and boundary-value problems (IVPs and BVPs) for systems of linear partial differential equations (PDEs) $\vec{u}_t = \sum_{i=1}^{m} B_i(x)\vec{u}_{x_i}$. The investigation is based on the rigorous computable analysis framework [2] and the classical computational complexity theory hierarchy

$$L \subseteq NC \subseteq P \subseteq NP \subseteq \sharp P \subseteq \sharp P^{\sharp P} \subseteq \ldots \subseteq PSPACE \subseteq EXP,$$

extended to the real setting [5]. We measure the complexity depending on the output precision parameter $n$, corresponding to computation of the output approximations with guaranteed precision $1/2^n$.

It is known that (a) ordinary differential equations (ODEs) with a polynomial/analytic right-hand side can be solved in PTIME [1]; (b) general non-linear ODEs with $C^1$-smooth right-hand part are optimally solved by Euler's Method in PSPACE [3], equivalently: in polynomial parallel time; (c) solving Poisson's linear PDEs corresponds to the complexity class $\sharp P$ [4].

Our main contributions are as follows:

**1.** Suppose the given IVP and BVP be *well posed* in that the classical solution $\vec{u} : [0;1] \times \overline{\Omega} \to \mathbb{R}$ (i) exists, (ii) is unique, and (iii) depends continuously on the initial function $\varphi = \vec{u}(0,x)$. More precisely we assume that $u(t,x) \in C^2$ and its $C^2$-norm is bounded linearly by $C^2$-norms of $\varphi(x)$ and $B_i(x)$ (in functional spaces guaranteeing all the required properties). Moreover suppose that the given IVP and BVP admit a (iv) stable and (v) approximating with at least the first order of accuracy (and approximation coefficient) explicit *difference scheme* represented by the $O(2^n) \times O(2^n)$ matrix $A_n$.

If $A_n$ and $\varphi$ are PTIME computable, then the solution function $\vec{u}$ belongs to the real complexity class PSPACE. If $A_n$ is additionally circulant of constant bandwidth (as for periodic BVPs), then $\vec{u}$ is in $\sharp P^{\sharp P}$; if $A_n$ is two-band (as for some IVPs), then $\vec{u}$ is in $\sharp P$.

This result generalizes [7, 4].

**2.** If the matrix coefficients $B_i(x)$ and initial function $\varphi(x)$ are, in addition *analytic* then their PTIME/PolyLogSPACE computability implies PTIME/PolyLogSPACE computability of the solution $\vec{u}(t, x)$.

This result generalizes the result of [1] about analytic ODEs.

As a main ingredient of the proofs we develop and analyze from the complexity viewpoint an efficient real polynomial/matrix/operator powering algorithm.

This is a joint work with Martin Ziegler, Ivan Koswara and Gleb Pogudin, partially published in [6].


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# Contributed Talks

# Overt choice

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We introduce and study the notion of *overt choice* for countably-based spaces and for CoPolish spaces. Overt choice is the task of producing a point in a closed set specified by what open sets intersect it. We show that the question of whether overt choice is continuous for a given space is related to topological completeness notions such as the Choquet-property; and to whether variants of Michael's selection theorem hold for that space. For spaces where overt choice is discontinuous it is interesting to explore the resulting Weihrauch degrees, which in turn are related to whether or not the space is Fréchet-Urysohn.

On the way, we suggest a definition of a computable quasi-Polish space and prove several characterizations, mirroring independent work by Hoyrup, Royas, Selivanov and Stull [4]. Some prior results regarding overt choice on computable metric spaces are due to Brattka and Presser [2], and to Brattka [1].

The full preprint is available as [3].

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* This project has received funding from the European Unions Horizon 2020 research and innovation programme under the Marie Sklodowska-Curie grant agreement No 731143, *Computing with Infinite Data*.
The first author was supported by JSPS Core-to-Core Program, A. Advanced Research Networks and by JSPS KAKENHI Grant Number 18K11166.

# DENSITY OF MAXIMAL COMPUTABILITY STRUCTURES

KONRAD BURNIK AND ZVONKO ILJAZOVIĆ

One way to impose computability on a metric space $(X,d)$ is to fix an effective separating sequence in $(X,d)$, i.e. a dense sequence $\alpha = (\alpha_i)$ in $(X,d)$ such that the distances $d(\alpha_i, \alpha_j)$ can be effectively computed. Then a sequence $(x_i)$ in $X$ is called computable if $(x_i)$ is computable with respect to $\alpha$. If we denote by $\mathscr{S}$ the set of all such sequences in $X$, then $\mathscr{S}$ has the following two properties:

(i) if $(x_i),(y_j) \in \mathscr{S}$, then the distances $d(x_i, y_j)$ can be effectively computed;
(ii) if $(x_i) \in \mathscr{S}$ and $(y_j)$ is a sequence in $X$ which is computable with respect to $(x_i)$, then $(y_j) \in \mathscr{S}$.

A more general way to impose computability on $(X,d)$ is the following. Let $\mathscr{S}$ be any set of sequences in $X$ which satisfies the properties (i) and (ii) above. We say that $\mathscr{S}$ is a computability structure on $(X,d)$ [7]. A sequence $(x_i)$ in $X$ is called computable if $(x_i) \in \mathscr{S}$ and a point $x \in X$ is called computable if $(x,x,x,\dots) \in \mathscr{S}$.

A computability structure on $(X,d)$ which consists of those sequences which are computable with respect to some fixed effective separating sequence $\alpha$ in $(X,d)$ is called separable [2]. Not every computable structure is separable, for example if $(X,d)$ is a metric space and $x \in X$, then $\{(x,x,x,\dots)\}$ is a computability structure on $(X,d)$ which is clearly not separable if $X$ has at least two points.

A computability structure $\mathscr{M}$ on $(X,d)$ is called maximal if there is no computability structure $\mathscr{S}$ on $(X,d)$ such that $\mathscr{M} \subseteq \mathscr{S}$ and $\mathscr{M} \neq \mathscr{S}$. Each separable computability structure is maximal, but a maximal computability structure need not be separable. For example, it is known that for each $a \in [0,1]$ there exists a unique maximal computability structure $\mathscr{M}_a$ on $[0,1]$ (with respect to the Euclidean metric) in which $a$ is a computable point, but $\mathscr{M}_a$ is not separable if $a$ is an incomputable real number [2].

A computability structure $\mathscr{S}$ on a metric space $(X,d)$ is called dense if the set of all computable points in $\mathscr{S}$ is dense in $(X,d)$. Each separable computability structure is clearly dense, so it is a dense maximal structure.

We consider the relationship between separable, maximal and dense maximal structures on subspaces of Euclidean space $\mathbb{R}^n$ (with respect to the Euclidean metric). In general, a maximal computability structure need not be dense. For example, if $X = [0,1] \cup \{2\}$, then there exists a maximal computability structure on $X$ which is not dense. Furthermore, if $X \subseteq \mathbb{R}^2$ is the boundary of a triangle, then $X$ has maximal computability structures which are not dense (note that such an $X$ is connected). We prove the following.

**Theorem 1.** *Let $X \subseteq \mathbb{R}^n$ be a convex set. Then each maximal computability structure on $X$ is dense.*

On the other hand, if $X \subseteq \mathbb{R}^n$ is a convex set, a dense maximal computability structure on $X$ need not be separable. Namely, the mentioned maximal computability structure $\mathscr{M}_a$ on $[0,1]$ is dense (for each $a \in [0,1]$) and it need not be separable. However, there are subspaces of $\mathbb{R}^n$ on which each dense maximal computability structure is separable, for

Supported by the Croatian Science Foundation under project 7459 CompStruct.

example the boundary of a triangle is such a space. Moreover, we have the following result. (By a sphere in $\mathbb{R}^n$ we mean a set of the form $\{y \in \mathbb{R}^n \mid d(y,x) = r\}$, where $x \in \mathbb{R}^n$ and $r > 0$ are fixed.)

**Theorem 2.** *Let $X \subseteq \mathbb{R}^n$ be a boundary of a simplex or a sphere. Then each dense maximal computability structure on $X$ is separable.*

Furthermore, if $X$ is a sphere with radius $r$, then $X$ has a separable computability structure if and only if $r$ is a computable number. Therefore, by Theorem 2, if $X$ is a sphere with an incomputable radius, none of the maximal computability structures on $X$ is dense.

In case $n = 2$ the result of Theorem 2 for spheres, i.e. circles, can be generalized in the following way. Let $X \subseteq \mathbb{R}^2$ be any conic (a hyperbola, a parabola or an ellipse). Then each dense maximal computability structure on $X$ is separable.

The statement of Theorem 2 does not hold for topological spheres (i.e. spaces which are homeomorphic to a sphere). We construct a topological circle $X$ in $\mathbb{R}^2$ and a dense maximal computability structure on $X$ which is not separable.

Finally, we consider certain polyhedra which are more general than the boundary of a triangle. Let $K$ be a nonempty finite simplicial complex of dimension 1 in $\mathbb{R}^n$, i.e. a family of line segments in $\mathbb{R}^n$ and their vertices such that for each two line segments $I, J \in K$ such that $I \neq J$ and $I \cap J \neq \emptyset$ we have $I \cap J = \{v\}$, where $v$ is a vertex of both $I$ and $J$. Suppose that each vertex of $K$ belongs to two distinct line segments of $K$. Let $X = \bigcup_{I \in K} I$. Then each dense maximal computability structure on $X$ is separable and there exists a maximal computability structure on $X$ which is not dense.

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# On representations of irrational numbers in subrecursive context

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**Abstract.** As is well known, there are many ways to represent irrational numbers: Dedekind cuts, Cauchy sequences, base $b$-expansions, etc. All of them induce the same class of computable real numbers relative full Turing computability. But when we restrict the notion of computability, we obtain quite an interesting interaction between the different representations. The first results in this direction concern the class of the primitive recursive functions. Specker proves in [6] that the set of real numbers with primitive recursive Dedekind cut is properly contained in the set of real numbers with primitive recursive decimal expansion, which in turn is a proper subset of the set of real numbers with primitive recursive Cauchy sequence (and primitive recursive modulus of convergence). Lehman constructs in [5] a real number, which does not have a primitive recursive Dedekind cut, but whose expansion in any base is primitive recursive. Similar results relative the class of functions, computable in polynomial-time can be found in Ko's paper [2].

Kristiansen in [3,4] has begun a systematic research of all known representations of irrational real numbers and many others. His results are formulated in broad generality with respect to arbitrary subrecursive classes, containing the elementary functions and closed under composition and bounded or unbounded primitive recursion (but not under unbounded search). Kristiansen introduces several new representations of irrational numbers by sum approximations. For example, the sum approximation from below in a fixed base $b$, in principle, enumerates the positions of all non-zero digits of the base-$b$ expansion of the real number. A symmetric representation is the sum approximation from above in base $b$. The links between these representations and the above-mentioned ones are rather complex. For example, combined with Dedekind cuts, the sum approximations from below and above give rise to seven different complexity classes of real numbers. In a recent paper [1] the present author, in collaboration with Kristiansen and Stephan, has studied some properties of general sum approximations, which are sum approximations uniform in the base.

The aim of the talk is to present a survey of known results on subrecursive representability of real numbers, as well as some new results, concerning the set of real numbers having a subrecursive expansion in some base and the set of real numbers having a subrecursive sum approximation in some base.

Let $\mathcal{S}$ be a sufficiently large natural subrecursive complexity class.

For any natural number $b \geq 2$ we denote by $\mathcal{S}_{bE}$ the set of all irrational numbers in $(0,1)$, whose expansion in base $b$ belongs to $\mathcal{S}$.

We also denote by $\mathcal{S}_C$ the set of all irrational numbers in $(0,1)$, which have a Cauchy sequence in $\mathcal{S}$.

A real number $\alpha$ is $\mathcal{E}^2$-irrational iff there exists a function $v \in \mathcal{E}^2$, such that $\left|\alpha - \frac{m}{n}\right| > \frac{1}{v(n)}$ for all integers $m, n$ with $n > 0$.

Our main result is the following

**Theorem.** *For any $\mathcal{E}^2$-irrational number $\alpha$ in $\mathcal{S}_C$ there exists an $\mathcal{E}^2$-irrational number $\beta$ in $\mathcal{S}_C$, such that $\alpha + \beta \notin \bigcup_{b \geq 2} \mathcal{S}_{bE}$.*

Now let $R$ any of the known representations, not equivalent to Cauchy sequences (thus $R$ might be continued fractions, sum approximations, etc.). Let $\mathcal{S}_R$ be the set of irrational numbers in $(0,1)$, which possess an $R$-representation, computable through functions from $\mathcal{S}$. Then we have

$$\{\ \alpha \in \mathcal{S}_C \mid \alpha \text{ is } \mathcal{E}^2 \text{-irrational}\ \} \quad \subseteq \quad \mathcal{S}_R \quad \subseteq \quad \bigcup_{b \geq 2} \mathcal{S}_{bE}$$

and it follows from the Theorem that $\mathcal{S}_R$ is not closed under addition.

Based on this observation we conclude that the Cauchy sequence representation is the only one suitable for subrecursive analysis.

**Keywords:** representations of irrational numbers, subrecursive classes, base expansions, sum approximations

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# Computability of topological entropy in the logistic family*

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In the theory of Dynamical Systems, topological entropy $h_{top}$ is a suitable tool to measure the dynamical complexity of a system, and its computability is a question of major interest [7]. In the case of symbolic dynamical systems, there is a fair amount of theory about this problem, ranging from non-computable examples [8] to more general characterizations for subshifts of finite type in one or more dimensions [6, 4].

For "non symbolic" systems, however, not much seems to be known. In the present work we consider the problem for the well known logistic family, given by

$$f_r(x) = rx(1-x), \quad \text{where } x \in [0,1], \quad \text{and } r \in [0,4] \text{ is the parameter.}$$

This family of systems exhibits a rich variety of behaviours as the parameter changes – from trivial to chaotic dynamics. Although there exist several numerical algorithms to estimate the topological entropy $h_{top}(f_r)$ given the parameter $r$, most of them rely on the so called *kneading sequence* [1], which is not known to be uniformly computable from the parameter [2].

Our main result is the following:

**Theorem**: *The topological entropy $h_{top}(f_r)$ in the logistic family is computable as a function of $r \in [0,4]$.*

The proof of this statement does not rely on the kneading sequence. Instead, we compute a sequence of parameters $r_i$ corresponding to the centres of

*This work is part of the author's undergraduate thesis under the supervision of professor Cristóbal Rojas (http://www.mat-unab.cl/~crojas/).

the hyperbolic components of the Mandelbrot set, together with their topological entropies, which can be computed using standard techniques involving subshifts of finite type associated to the system. Next, using dynamical properties of hyperbolic components we describe a procedure to enumerate a sequence of dyadic parameters $\{d_{k_i}\}$ which is dense in these components, and for which we can compute the topological entropy. This construction is possible by the means of the Main Theorem of Section 5.3 in [3]. Taking advantage of the density of hyperbolic components in the space of parameters [5], we describe an algorithm that, provided with arbitrarily good approximations of a given parameter $r \in [0, 1]$, outputs arbitrarily good approximations of the value $h_{top}(f_r)$.

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# A WEAKLY COMPUTABLE NUMBER WHICH CAN BE ONLY DECOMPOSED INTO RANDOM LEFT-COMPUTABLE NUMBERS

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In this paper we are concerned with left-computable real numbers and with weakly computable real numbers. A real number is called *left-computable* if there exists a computable increasing sequence of rational numbers converging to it. The set of the left-computable numbers is a widely investigated real number class and of particular interest both to computable analysis and to algorithmic information theory. For left-computable real numbers there is a natural reducibility relation due to Solovay [9], which leads to a classification of left-computable real numbers according to their Solovay degrees.

**Definition 1** ([9])**.** Let $x$ and $y$ be left-computable numbers. We say that $x$ is *Solovay-reducible* to $y$ (denoted by $x \leq_S y$) if there exist computable increasing sequences $(x_n)_n$ and $(y_n)_n$ of rational numbers converging to $x$ and $y$, respectively, and a constant $c > 0$ such that for all $n \in \mathbb{N}$ we have:

$$x - x_n < c \cdot (y - y_n)$$

It is obvious that the Solovay reduction is reflexive, and it is easy to see that it is transitive. As usual, one defines an equivalence relation $\equiv_S$ on the left-computable numbers by $x \equiv_S y$ if, and only if, $x \leq_S y$ and $y \leq_S x$. Its equivalence classes are called *Solovay degrees*. There is a smallest Solovay degree, the set of computable numbers [2]. And there is a largest Solovay degree whose elements can be described in several different ways.

**Proposition 2** ([3, 9, 2, 6])**.** *For a left-computable number $x$ the following are equivalent:*

(1) *For all left-computable numbers $y$ we have $y \leq_S x$.*
(2) *The fractional part of $x$ (that is, the unique number $x' \in [0, 1[$ with $x - x' \in \mathbb{Z}$) is an Omega number in the sense of Chaitin* [3]*.*
(3) *$x$ is Martin-Löf random* [7]*.*

Furthermore $\leq_S$ is an upper semilattice: The $\leq_S$-supremum of the Solovay degrees of two left-computable numbers $x$ and $y$ is the Solovay degree of the left-computable number $x + y$ [2]. For further results on Solovay degrees see [5] and the monograph [4].
The left-computable numbers do not form a field, but there exists a smallest field containing all left-computable numbers. This field ist the set of the weakly computable numbers, which were introduced by Ambos-Spies, Weihrauch and Zheng [1].

**Definition 3** ([1])**.** A real number $z$ is called *weakly computable* if there exist left-computable numbers $x$ and $y$ with $z = x - y$.

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If one wishes to understand the computability-theoretic complexity of a weakly computable real number $z$ one may ask for the Solovay degrees of left-computable numbers $x, y$ with $z = x - y$. Any weakly computable number $z$ can be written as the difference of two left-computable numbers in the largest Solovay degree of left-computable numbers: If $z = x - y$ for two left-computable numbers $x, y$ and if $\Omega$ is a random left-computable number, then the numbers $x' := x + \Omega$ and $y' := y + \Omega$ are also random left-computable numbers and they satisfy $z = x' - y'$. One may now go into the other direction and ask whether any weakly computable number can be written as the difference of two left-computable numbers that are "easy" with respect to the Solovay reduction. The following theorem gives a negative answer to this question.

**Theorem 4.** *There exists a weakly computable number $z$ such that for all left-computable numbers $x$ and $y$ with $z = x - y$ both $x$ and $y$ are random.*

In other words, there exists a weakly computable number $z$ such that, for all left-computable numbers $x$ and $y$, if $z = x - y$ then $x$ and $y$ are elements of the largest Solovay degree. This theorem is the main result of this paper. The proof is by an infinite injury priority argument.

In the context of this result the following observation by Rettinger and Zheng [8] is interesting: For every random weakly computable number $z$ either $z$ or $-z$ is left-computable. If, for example, $z$ is random and left-computable and $z = x - y$ is a decomposition of $z$ into two left-computable numbers $x$ and $y$, then $x = y + z$ must be random as well while $y$ can even be chosen to be equal to 0. So a weakly computable number satisfying the property in Theorem 4 cannot be random itself.

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# COMPUTABILITY OF SPACES WITH ATTACHED ARCS

ZVONKO ILJAZOVIĆ AND MATEA JELIĆ

A compact subset $S$ of Euclidean space $\mathbb{R}^n$ is computable if $S$ can be effectively approximated by finitely many rational points with any given precision. More general than computable are semicomputable sets: a compact set $S \subseteq \mathbb{R}^n$ is semicomputable if $S = f^{-1}(\{0\})$ for some computable function $f : \mathbb{R}^n \to \mathbb{R}$.

In general, a semicomputable set need not be computable. For example, there exists a line segment in $\mathbb{R}$ (or in any $\mathbb{R}^n$) which is semicomputable but not computable. Thus a semicomputable arc (i.e. a semicomputable set homeomorphic to $[0,1]$) need not be computable. However, if a semicomputable arc has computable endpoints, then it must be computable. It is also known that any semicomputable topological circle must be computable.

In a computable metric space, which is a more general ambient space than Euclidean space, the notions of a computable and a semicomputable set can also be defined and the general question is: under what conditions is a semicomputable set in a computable metric space computable? The results for arcs and topological circles also hold in computable metric spaces and the following definition arises. We say that a topological space $\Delta$ has computable type if for any computable metric space $(X,d,\alpha)$ and any topological embedding $f : \Delta \to X$ the following implication holds:

$$f(\Delta) \text{ semicomputable } \Rightarrow f(\Delta) \text{ computable.}$$

For example, each circle has computable type. Moreover, for each $n \in \mathbb{N} \setminus \{0\}$ the unit sphere in $\mathbb{R}^n$ has computable type. In fact, a more general result holds: if $M$ is a compact manifold, then $M$ has computable type.

On the other hand, $[0,1]$ does not have computable type. However, if $(X,d,\alpha)$ is a computable metric space and $f : [0,1] \to X$ an embedding such that $f([0,1])$ and $\{f(0), f(1)\}$ are semicomputable sets, then it easily follows that $f(0)$ and $f(1)$ are computable points and the previously mentioned result implies that $f([0,1])$ is computable. So the following definition arises.

Let $\Delta$ be a topological space and let $\Sigma$ be a subspace of $\Delta$. We say that the pair $(\Delta,\Sigma)$ has computable type if for any computable metric space $(X,d,\alpha)$ and any topological embedding $f : \Delta \to X$ the following implication holds:

$$f(\Delta) \text{ and } f(\Sigma) \text{ are semicomputable } \Rightarrow f(\Delta) \text{ computable.}$$

We have that $([0,1],\{0,1\})$ has computable type. It is also known that $(\mathbb{B}^n,\mathbb{S}^{n-1})$ has computable type for each $n \in \mathbb{N} \setminus \{0\}$, where $\mathbb{B}^n$ is the unit closed ball and $\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n$. Moreover, if $M$ is a compact manifold with boundary, then $(M,\partial M)$ has computable type. Note that a topological space $\Delta$ has computable type if and only if the topological pair $(\Delta,\emptyset)$ has computable type.

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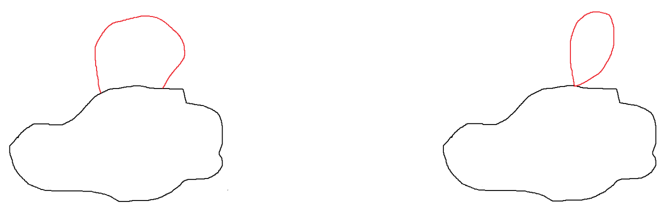
Supported by the Croatian Science Foundation under project 7459 CompStruct.

We consider the following general question: if $(\Delta_1,\Sigma_1)$ and $(\Delta_2,\Sigma_2)$ have computable types and a topological pair $(\Delta,\Sigma)$ is obtained from $(\Delta_1,\Sigma_1)$ and $(\Delta_2,\Sigma_2)$ by a certain topological construction, does $(\Delta,\Sigma)$ have computable type?

For example, it is not hard to prove the following result: if $(\Delta_1,\Sigma_1)$ and $(\Delta_2,\Sigma_2)$ have computable types, then $(\Delta_1 \sqcup \Delta_2, \Sigma_1 \sqcup \Sigma_2)$ has computable type, where $X \sqcup Y$ denotes the disjoint union of topological spaces $X$ and $Y$.

Let $X$ and $Y$ be topological spaces, let $A$ be a subspace of $Y$ and let $f : A \to X$ be a continuous function. Then the space $X \cup_f Y$ (so called adjunction space) is defined as a quotient space obtained from $X \sqcup Y$ by identifying $a$ and $f(a)$ for each $a \in A$.

We examine adjunction spaces in the context of computable type. In particular, we examine spaces $\Delta \cup_f [0,1]$, where $\Delta$ is a topological space and $f : \{0,1\} \to \Delta$. We imagine $\Delta \cup_f [0,1]$ as a space which is obtained by gluing an arc to $\Delta$ along its endpoints. The figures below illustrate two possibilities how a red arc can be glued to $\Delta$ along its endpoints.



We have the following result.

**Theorem 1.** *Let $\Delta$ be topological space which has computable type. Let $f : \{0,1\} \to \Delta$ be a function. Then the pair $(\Delta \cup_f [0,1], \Delta)$ has computable type.*

Note that in the previous theorem $\Delta$ is considered as a subspace of $\Delta \cup_f [0,1]$ in the obvious way. Note also that, under the assumption of the theorem, the topological space $\Delta \cup_f [0,1]$ need not have computable type. Namely, let $\Delta = \{0,1\}$ and $f : \{0,1\} \to \{0,1\}$, $f(0) = 0$, $f(1) = 1$. Then $\Delta$ has computable type, but $\Delta \cup_f [0,1]$ does not have computable type since $\Delta \cup_f [0,1]$ is homeomorphic to $[0,1]$.

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**On Subrecursive Representability of Irrational Numbers: Continued Fractions and Contraction maps**

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There are numerous ways to represent real numbers. We may use, e.g., Cauchy sequences, Dedekind cuts, numerical base-10 expansions, numerical base-2 expansions and continued fractions. If we work with full Turing computability, all these representations yield the same class of computable real numbers. If we work with some restricted notion of computability, e.g., polynomial time computability or primitive recursive computability, they do not. If we are not allowed to carry out unbounded search, rather surprising and unexpected situations occur when we try to convert irrationals from one representation to another. This phenomenon has been investigated over the last seven decades by Specker [6], Mostowski [7], Lehman [8], Ko [9, 10], Kristiansen, Georgiev, Stephan [1–5] and quite a few more.

We will survey some of the results published in [1–3]. Thereafter we will present some new results on contraction maps, continued fractions and best approximations.

We restrict our attention to irrationals between 0 and 1. When we say that $\alpha$ is irrational, we mean that $\alpha$ an irrational between 0 and 1.

*Contractions Maps.* Contraction maps are known from the theory of metric spaces. We say that a function $F$ is a *contraction map* if we have

$$|F(q_1) - F(q_2)| \ < \ |q_1 - q_2|$$

for any rationals $q_1, q_2$ where $q_1 \neq q_2$. For any contraction map $F$ there exists a unique irrational number $\alpha$ such that we have $|\alpha - q| > |\alpha - F(q)|$ for any rational $q$. We say that $F$ *represents* that $\alpha$.

*Continued Fractions.* Continued fractions are well known from the literature. For any irrational $\alpha$ there is a continued fraction $[\ 0\ ; a_1, a_2, \dots\ ]$, where each $a_i$ is a positive integer, such that

$$\alpha \ = \ 0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_2 + \dots}}}$$

*Best Approximations.* Let $a$ and $b$ be relatively prime natural numbers. The fraction $a/b$ is a *left best approximant* of $\alpha$ if $c/d \leq a/b < \alpha$ or $\alpha < c/d$ for any natural numbers $c, d$ where $d \leq b$. The fraction $a/b$ is a *right best approximant* of $\alpha$ if $\alpha < a/b \leq c/d$ or $c/d < \alpha$ for any natural numbers $c, d$ where $d \leq b$. A *left best approximation* of $\alpha$ is a strictly increasing sequence of fractions $(a_n/b_n)_{n\in\mathbb{N}}$ such that $a_0/b_0 = 0/1$ and each $a_n/b_n$ is a left best approximant to $\alpha$. A *right best approximation* of $\alpha$ is a strictly decreasing sequence of fractions $(a_n/b_n)_{n\in\mathbb{N}}$ such that $a_0/b_0 = 1/1$ and each $a_n/b_n$ is a right best approximant to $\alpha$.

*Classes of Irrational Numbers.* Let $\mathcal{S}$ be a subrecursive class of functions which is closed under primitive recursive operations. Let

- $\mathcal{S}_{[\,]}$ denote the class of irrational that have continued fractions in $\mathcal{S}$
- $\mathcal{S}_F$ denote the class of irrationals that have contraction maps in $\mathcal{S}$
- $\mathcal{S}_<$ denote the class of irrationals that have left best approximations in $\mathcal{S}$
- $\mathcal{S}_>$ denote the class of irrationals that have right best approximations in $\mathcal{S}$
- $\mathcal{S}_D$ denote the class of irrationals that have Dedekind cuts in $\mathcal{S}$.

*Main Results.* We have

$$\mathcal{S}_< \cap \mathcal{S}_> = \mathcal{S}_F = \mathcal{S}_{[\,]} \quad \text{and} \quad \mathcal{S}_< \not\subseteq \mathcal{S}_> \quad \text{and} \quad \mathcal{S}_> \not\subseteq \mathcal{S}_< \quad \text{and} \quad \mathcal{S}_< \cup \mathcal{S}_> \subset \mathcal{S}_D \; .$$

Moreover, $\mathcal{S}_< = \mathcal{S}_{g\uparrow}$ and $\mathcal{S}_> = \mathcal{S}_{g\downarrow}$ where $\mathcal{S}_{g\uparrow}$ ($\mathcal{S}_{g\downarrow}$) is the class of irrationals that have general sum approximations from below (above) in $\mathcal{S}$ (general sum approximations are defined and explained in [3] and [1]).

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# Is Brownian Motion Computable?*

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1D *Brownian Motion* (aka Wiener Process) is a the space $\mathcal{C}_0[0;1]$ of continuous functions $f : [0;1] \to \mathbb{R}$ s.t. $f(0) = 0$, equipped with Wiener measure $\mu$. We approach the question of whether Wiener process is computable [DF13].

**Definition 1.** *Let* $\mathcal{C} := \{0,1\}^\omega$ *denote Cantor space, equipped with the canonical Borel probability measure* $\operatorname{vol}(\vec{w} \circ \mathcal{C}) = 2^{-|\vec{w}|}$. *Let* $X$ *denote a topological space with representation* $\xi :\subseteq \mathcal{C} \twoheadrightarrow X$ *and Borel probability measure* $\mu$.
*We say that* $F :\subseteq \mathcal{C} \to \operatorname{dom}(\xi)$ *is a* $\xi$-realizer *of (a random variable with distribution)* $\mu$ *if* $\mu(S) = \operatorname{vol}\Big(F^{-1}\big[\xi^{-1}[S]\big]\Big)$ *holds for every Borel* $S \subseteq X$. *Call* $\mu$ computable *if it has a computable realizer.*

Note that $\operatorname{dom}(F)$ must be Borel of measure 1. A $\xi$-realizer amounts to a *strong probabilistic name* with respect to the 'fair' 50:50 *probabilistic process* in the sense of [SS06, §3]. From the perspective of a random variable as a function, our terminology agrees with [Wei00, §3.1].

**Example 2 (Real Unit Interval)** *Let* $\rho_{\mathrm{b}} : \mathcal{C} \ni \bar{b} \mapsto \sum_{j\geq 0} b_j 2^{-j-1}$ *denote the continuous (but not admissible)* binary *representation of the real unit interval* $[0;1]$. *Then the identity is a computable realizer of the Lebesgues measure on* $[0;1]$ *w.r.t.* $\rho_{\mathrm{b}}$.

Recall that a sequence $R_n :\subseteq \mathcal{C} \to X$ of random variables converges *almost surely* to $R :\subseteq \mathcal{C} \to X$ if the set $\big\{\bar{u} : R_n(\bar{u}) \to R(\bar{u})\big\} \subseteq \mathcal{C}$ has measure 1.

On the other hand for $(X,d)$ a metric space, *uniform* almost sure convergence of $R_n$ to $R$ means that there exists $U \subseteq \operatorname{dom}(R) \cap \bigcap_n \operatorname{dom}(R_n)$ of measure 1 such that $\sup_{\bar{u}\in U} d\big(R_n(\bar{u}), R(\bar{u})\big) \to 0$.

**Lemma 3.** *Suppose* $U \subseteq \mathcal{C}$ *has measure 1 and* $R_n : U \to X$ *is a* $\xi$*-computable sequence converging* effectively *uniformly almost surely to* $R : U \to X$ *in the sense that there exists a* recursive $\nu : \mathbb{N} \to \mathbb{N}$ *such that*

$$\forall n \geq \nu(m) : \quad \sup_{\bar{u}\in U} d\big(R_n(\bar{u}), R(\bar{u})\big) \leq 2^{-m} \quad .$$

*Then* $R$ *is almost surely* $\xi$*-computable.*

For other notions of randomized computability see [Bos08,BGH15]. A *computable* realizer $R$ of measure space $(X,\mu)$ must be almost surely computable.

*Supported by the National Research Foundation of Korea (grant NRF-2017R1E1A1A03071032) and the International Research & Development Program of the Korean Ministry of Science and ICT (grant NRF-2016K1A3A7A03950702).

**Example 4 (Wiener Process)** *Let $X := \mathcal{C}([0;1],\mathbb{R})$ denote the space of continuous real functions $W = \{W_t\}_{t\in\mathbb{R}}$ on the unit interval, equipped with the uniform metric and the Wiener Measure. Among its well-known implicit 'representations' in mathematics, we report the following:*

*a) Let $\varphi_0(t) = t$ and*

$$\varphi_{n,j}(t) = \begin{cases} 2^{(n-1)/2}\cdot(t-\frac{k-1}{2^n}) & \frac{k-1}{2^n}\le t\le\frac{k}{2^n} \\ 2^{(n-1)/2}\cdot(\frac{k+1}{2^n}-t) & \frac{k}{2^n}\le t\le\frac{k+1}{2^n} \\ 0 & \textit{otherwise} \end{cases}, \qquad 0\le k<2j, \quad 1\le j\le 2^{n-1}$$

*denote the* Schauder *'hat' functions and $R_{n,j}$ independent standard normally distributed random variables. Then following sequence converges to the Wiener process almost surely:*

$$W_t^N(\omega) = R_0(\omega)t + \sum_{n=1}^{N}\sum_{j=1}^{2^{n-1}} R_{n,j}(\omega)\varphi_{n,j}(t) \tag{1}$$

*b) Let $R_i$ be independent standard normally distributed random variables. Then following sequence converges to the Wiener process in $L_2$ metric:*

$$W_t^N(\omega) = \sqrt{2}\sum_{i=1}^{N} R_i\frac{\sin{(k-\frac{1}{2})\pi t}}{(k-\frac{1}{2})\pi} \tag{2}$$

*c) Let $(X_i)_{i\in\mathbb{N}}$ be random variable with mean 0 and variance 1 and $S_n = \sum_{i=1}^{n} X_i$ Then following sequence converges to Wiener process in distribution:*

$$W_t^N(\omega) = \frac{S_{\lfloor Nt\rfloor}}{\sqrt{N}} \tag{3}$$

Item (a) is mentioned in [Col15, §6]. None of the sequences in (a), (b), and (c) converges *uniformly* almost surely; hence Lemma 3 does not apply.

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# A COMPUTABILITY-THEORETIC PROOF OF LUSIN'S THEOREM

RUSSELL MILLER

Abstract. Lusin's Theorem states that, for every Borel-measurable function $\mathbf{f}$ on $\mathbb{R}$ and every $\epsilon > 0$, there exists a continuous function $\mathbf{g}$ on $\mathbb{R}$ which is equal to $\mathbf{f}$ except on a set of measure $< \epsilon$. We give a proof of this theorem using computability theory, relating it to the near-uniformity of the Turing jump operator. From the proof we derive results on the extent to which $\mathbf{g}$ can be produced effectively from an oracle and a Turing program computing $\mathbf{f}$.

Lusin's Theorem is a standard result in a first course in real analysis. It describes the extent to which a "reasonably nice" function on the real numbers $\mathbb{R}$ (formally, a Borel-measurable function) can fail to be continuous. Along with related results, it is frequently listed as one of Littlewood's Three Principles.

**Theorem 1** (Lusin's Theorem, 1912)**.** *For every Borel-measurable function* $\mathbf{f} : \mathbb{R} \to \mathbb{R}$ *and every* $\epsilon > 0$*, there exists a continuous function* $\mathbf{g} : \mathbb{R} \to \mathbb{R}$ *such that*

$$\mu(\{\mathbf{x} \in \mathbb{R} : \mathbf{f}(\mathbf{x}) \neq \mathbf{g}(\mathbf{x})\}) < \epsilon.$$

Alternative versions allow $\pm\infty$ as values of the functions in question. Lusin's Theorem is the best possible result in this direction, as there do exist measurable functions $\mathbf{f}$ such that, for every continuous $\mathbf{g}$, $\mu(\{\mathbf{x} \in \mathbb{R} : \mathbf{f}(\mathbf{x}) \neq \mathbf{g}(\mathbf{x})\}) > 0$.

In computable analysis, the Borel-measurable functions $\mathbf{f} : \mathbb{R} \to \mathbb{R}$ are precisely those that can be computed by a Turing functional $\Phi$, using an oracle $(S \oplus X)^{(\alpha)}$, where $S \subseteq \omega$ is a fixed oracle set, $\alpha$ is a fixed countable ordinal, and $X$ is any Cauchy sequence that converges effectively to its limit $\mathbf{x} \in \mathbb{R}$. Defining the $\alpha$-th jump $(S \oplus X)^{(\alpha)}$ requires a fixed presentation of the ordinal $\alpha$, which in turn may require an oracle to compute that presentation, as not all countable ordinals have computable presentations. One may fold this oracle into the set $S$, making it available to the functional in order to work with a single presentation of $\alpha$ and thus to define the $\alpha$-th jump uniformly on all subsets of $\omega$. The upshot is that

$$\Phi^{S \oplus (S \oplus X)^{(\alpha)}} : \omega \to \mathbb{Q}$$

The author was partially supported by grant # DMS – 1362206 from the National Science Foundation by grant # 581896 from the Simons Foundation, and by several grants from the PSC-CUNY Research Award Program. This work was initiated at a workshop held at the Schloß Dagstuhl in February 2017, where Vasco Brattka aided its development with useful conversations.

is required to be a total function, outputting a sequence of rational numbers $q_0, q_1, \ldots$ that converges effectively to the value $\mathbf{f}(\mathbf{x})$. This sequence may depend on the choice of the input sequence $X$, but its limit $\mathbf{f}(\mathbf{x})$ must depend only on the limit $\mathbf{x}$ of the input sequence.

It is well known that, for each fixed $\alpha$ and for each $\epsilon > 0$, the $\alpha$-th jump $A^{(\alpha)}$ (again using a fixed $S$-computable presentation of $\alpha$) satisfies $A^{(\alpha)} \leq_T S \oplus \varnothing^{(\alpha)} \oplus A$ for all $A \subseteq \omega$ outside a set of measure 0. The reduction is not uniform, but it is arbitrarily close to being so: there is an $S$-computable function $h : \omega \to \omega$ such that, for every rational $\epsilon > 0$,

$$\mu(\{A \subseteq \omega : \Upsilon_\epsilon^{S\oplus\varnothing^{(\alpha)}\oplus A} \neq A^{(\alpha)}\}) < \epsilon,$$

where $\Upsilon_\epsilon$ is the $h(\epsilon)$-th Turing functional $\Phi_{h(\epsilon)}$.

The goal of this presentation is to use this near-uniformity of the jump operator to present a proof of Lusin's Theorem. The basic idea is simple: given indices for a function $\Phi^{S\oplus(S\oplus X)^{(\alpha)}}$ computing a Borel-measurable $\mathbf{f} : \mathbb{R} \to \mathbb{R}$, we wish to produce a Turing functional $\Theta$ such that $\Theta^{S^{(\alpha)}\oplus X}$ computes a function $\mathbf{g}$ which instantiates Lusin's Theorem for the given $\mathbf{f}$. The idea is to use $\Upsilon_\epsilon$ to compute $(S \oplus X)^{(\alpha)}$ from $\Theta$'s oracle, and then to run the computation of $\Phi$ with the oracle $(S \oplus X)^{(\alpha)}$. For all but $\epsilon$-many inputs $X$, this will indeed output a Cauchy sequence converging to the correct value. The challenge is to ensure that, on the "bad" inputs, the computation does not go too far astray: if $X$ and $\widetilde{X}$ converge to the same limit $\mathbf{x}$, but one or both are bad inputs, we need $\Theta^{S^\alpha\oplus X}$ and $\Theta^{S^\alpha\oplus\widetilde{X}}$ to compute Cauchy sequences that both converge fast to the same limit. (If either one, say $X$, is a good input, with $X^{(\alpha)} = \Upsilon_\epsilon^{\varnothing^{(\alpha)}\oplus X}$, then this limit will in fact be $\mathbf{f}(\mathbf{x})$, where $\mathbf{x}$ is the limit of $X$. If not, then this is an $\mathbf{x}$ for which we allow $\mathbf{g}(\mathbf{x}) \neq \mathbf{f}(\mathbf{x})$, but we must still get the same value $\mathbf{g}(\mathbf{x})$ from every input sequence converging fast to $\mathbf{x}$.) We intend to show that this can be done: the key is the ability to enumerate a set of intervals, of total measure $< \epsilon$, containing all of the bad inputs. The resulting $\mathbf{g}$ will then be continuous, of course, so will prove Lusin's Theorem. Moreover, we will examine the extent to which $\Theta$ and its oracle set arise uniformly from the given $S$, $\alpha$, and $\Phi$.

The speaker is a computable structure theorist by trade, making his first real foray into computable analysis, and will be grateful for any references or suggestions about ways to improve the proof he presents.

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# Effective local compactness and the hyperspace of located sets

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We revisit the question of how to effectivize the notion of local compactness. A preprint is available as [4]. There have been previous studies of effective local compactness ([6, 5, 2]), albeit restricted to computable Polish spaces. We compare the definitions and show that they are equivalent. There are some subtleties involved, which could be interpreted as demonstrating that the previous definitions were (even for computable Polish spaces) prima facie too restrictive (as for some examples establishing their effective local compactness would be more work than it should be).

For non-Hausdorff spaces, there are several competing (and non-equivalent) definitions of local compactness. We will effective the existence of a compact local neighborhood basis for our purposes:

**Definition 1.** We call a represented space $\mathbf{X}$ *effectively locally compact*, if the map

$$\text{CompactBase} :\subseteq \mathbf{X} \times \mathcal{O}(\mathbf{X}) \rightrightarrows \mathcal{O}(\mathbf{X}) \times \mathcal{K}(\mathbf{X})$$

with $\text{dom}(\text{CompactBase}) = \{(x, U) \mid x \in U\}$ and $(V, K) \in \text{CompactBase}(x, U)$ iff $x \in V \subseteq K \subseteq U$ is computable.

In countably-based spaces, we can ask for a specific structure that witnesses effective local compactness. Manipulating this structure will be how we prove further results.

**Definition 2.** Let an effective relatively compact system (ercs) of a represented space be a triple $((U_n)_{n\in\mathbb{N}}, (B_n)_{n\in\mathbb{N}}, R)$ where

1. $(U_n \in \mathcal{O}(\mathbf{X}))_{n\in\mathbb{N}}$ is a computable sequence of open sets;
2. $(B_n \in \mathcal{K}(\mathbb{N}))_{n\in\mathbb{N}}$ is a computable sequence of compact sets;
3. and $R \subseteq \mathbb{N}\times\mathbb{N}$ is a computably enumerable relation such that $(m, n) \in R$ implies $U_m \subseteq B_n$;

such that for any open set $U \in \mathcal{O}(\mathbf{X})$ it holds that:

$$U = \bigcup_{\{n \mid U \supseteq B_n\}} \bigcup_{\{m \mid (m,n)\in R\}} U_m$$

The idea is that $R$ codes a formal containment relation between the enumerated open and compact sets. We shall write $U_n \ll B_m$ for $(n, m) \in R$.

We briefly explore how admitting an ercs, being compact, and being computably compact are related:

**Proposition 3.** Let $\mathbf{X}$ admit an ercs and be compact. Then $\mathbf{X}$ is computably compact.

If a space admits an ercs and is computably Hausdorff, it is already computably metrizable. This follows very directly from Schröder's effective metrization theorem [1]. The latter states that computably regular effectively countably-based spaces are computably metrizable. Their formulation of being computably regular actually takes the very same form as the definition of ercs, except that closed sets are used in the place of compact sets. Since being computably Hausdorff suffices to translate from compact sets to closed sets, it follows that a computably Hausdorff space admitting an ercs is already computably regular.

In computable metric spaces, we can be more specific regarding how the sets $B_n$ in ercs look like; namely, we can demand that the compact sets be closed balls:

**Proposition 4.** Let $(\mathbf{X}, d)$ be a computable metric space. Then the following are equivalent:

1. $\mathbf{X}$ admits an ercs.
2. The map $\mathrm{CompactBall} : \mathbf{X} \rightrightarrows (\mathbb{N} \times \mathcal{K}(\mathbf{X}))$ where $(n, K) \in \mathrm{CompactBall}(x)$ iff $K = \overline{B}(x, 2^{-n})$ is well-defined and computable.

**Corollary 5.** Every computably compact computable metric space admits an ercs.

As an application of the machinery of effective local compactness. We study the hyperspace $(\mathcal{A} \wedge \mathcal{V})(\mathbf{X})$ of sets given as both closed and overt. In the language of Weihrauch, this is the full information representation of the closed sets. In constructive mathematics, the computable elements of $(\mathcal{A} \wedge \mathcal{V})(\mathbf{X})$ are often called *located*.

Our main result is that whenever $\mathbf{X}$ admits an ercs, then $(\mathcal{A} \wedge \mathcal{V})(\mathbf{X})$ is computably compact and computably metrizable. This generalizes a result from [3] for computably compact computable metric spaces.

## Acknowledgements

This research was partially supported by the Royal Society International Exchange Grant 170051 "Continuous Team Semantics: On dependence and independence in a continuous world".

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# Convergence Dominated Reducibility and Divergence Bounded Computable Real numbers

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## (Abstract)

A real number is called *c.a.* (computably approximable) if it is the limit of a computable sequence of rational numbers. To compare the (non-)computability of the c.a. reals, one possible way is to check how quickly they can be approximated by the computable sequences of rational numbers. Solovay [7] introduced a reducibility notion (so called Solovay reducibility) by a direct comparison of the speeds of convergence of the increasing computable sequences when he investigated the relative randomness of the c.e. real numbers. We call a real *c.e. (computably enumerable)* if it is the limit of an increasing computable sequences of rational numbers. A c.e. real $x$ is called *Solovay reducible to* another c.e. real number $y$ (denoted by $x \leq_S^0 y$) if there are two computable sequences of rational numbers $(x_s)$ and $(y_s)$ increasingly converging to $x$ and $y$, respectively, and a constant $c$ such that

$$(x - x_s) \leq c \cdot (y - y_s) \tag{1}$$

for all natural numbers $s$.

Randomness reflects a kind of non-computability. The Solovay reducibility classifies somehow different levels of (non)-computability of c.e. real numbers. Solovay reducibility characterizes the randomness of c.e reals perfectly as well: if $x \leq_S^0 y$, then $K(x \upharpoonright n) \leq K(y \upharpoonright n) + O(1)$, where $K(\alpha)$ denotes the Kolmogorov complexity of the string $\alpha$. Furthermore, a c.e. real number is random iff it is Solovay-complete and iff it is an $\Omega$-number (cf. [7, 2, 5]). Thus, the Solovay reduction is really proper way of classifying the randomness (or non-computability) levels of the c.e. real numbers.

The Solovay reducibility can be extended straightforwardly to the class of all c.a. reals by replacing the inequality (1) with the following one

$$|x - x_s| \leq c \cdot |y - y_s|. \tag{2}$$

This works for all convergent sequences $(x_s)$ and $(y_s)$. However, this extended reducibility doesn't match the intuition of computability (or randomness) classification at all. For example, for any non-rational computable real number $x$, there is a d-c.e real number $y$ such that $x$ is not reducible to $y$ in the sense of (2), where d-c.e. reals are the differences of c.e. reals. Thus, the reduction defined in this way is not even transitive, because $x$ is reducible to any rational number $z$ and $z$ is also reducible to the d-c.e. real number $y$.

A slightly modified version of Solovay reduction is proposed by Zheng and Rettinger in [9] by replacing the inequality (2) with following condition

$$|x - x_s| \leq c \cdot \left(|y - y_s| + 2^{-s}\right). \tag{3}$$

We denote this reduction by $x \leq_S^1 y$. This definition coincides with the original Solovay reduction on the c.e. real numbers and works perfectly well on the class of d-c.e. real numbers. For example, it is shown in [8] that a real number $x$ is d-c.e. if and only if it is reducible to a random c.e. number in this sense. The condition (3) requires essentially that the approximation errors of $|x - x_s|$ are bounded by a fixed linear combination of the approximation errors of $|y - y_s|$. To replace the "linear combination domination" by a more general "computable domination", we introduce the following definition.

**Definition 0.1** *A real number $x$ is* convergence-dominated reducible *(cd-reducible, for short) to $y$ (denoted by $x \leq_{cd} y$) if there exist a monotone total computable real function $h : \mathbb{R} \to \mathbb{R}$ with $h(0) = 0$ and two computable sequences $(x_s)$ and $(y_s)$ of rational numbers which converges to $x$ and $y$, respectively, such that*

$$\forall s \in \mathbb{N}(|x - x_s| \leq h(|y - y_s|) + 2^{-s}). \tag{4}$$

The cd-reduction is very closely related to the class or *divergence bounded computable (d.b.c in short)* reals, where a real number $x$ is called d.b.c if there is a computable sequence $(x_s)$ of rational numbers which converges to $x$ and a computable function $h$ such that, for any $n$, the number of non-overlapping index pair $(i, j)$ with the condition $|x_i - x_j| \geq 2^{-n}$ is bounded by $h(n)$. We will show that, a real number is d.b.c if and only if it is cd-reducible to a c.e. random real. That is,

**Theorem 0.2** *A computably approximable real number is d.b.c. iff it is cd-reducible to a c.e. random real number.*

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# On maximal Co-Polish Spaces

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April 29, 2019

Co-Polish spaces play an important role in Type Two Complexity Theory. They allow for "Simple Complexity Theory". This means that one can measure time complexity for functions on Co-Polish spaces in terms of a *discrete* (rather than a continuous) parameter on the input and the desired output precision. Note that for general spaces, e.g. for non-locally-compact metric spaces like $\mathcal{C}(\mathbb{R}, \mathbb{R})$, an indiscrete parameter on the input is necessary, as seen in the approach by A. Kawamura and S. Cook (see [3]).

**Definition 1** A topological space $\mathsf{X}$ is called *Co-Polish*, if $\mathsf{X}$ is the direct limit of an increasing sequence of compact metrisable subspaces.

Basic examples of Co-Polish spaces are separable locally compact metric spaces. The space of analytic functions on $[0;1]$ yields an example of a Co-Polish space that is not metrisable (cf. [4]). The name "Co-Polish" is motivated by the fact that Co-Polish spaces are exactly those regular qcb-spaces for which the lattice of opens endowed with the Scott topology is quasi-Polish in the sense of M. de Brecht (cf. [1]). Moreover, Co-Polish spaces are characterised as those Hausdorff qcb-spaces that have an admissible representation with a locally compact domain (see [5]).

### Maximal Co-Polish spaces

We call a Co-Polish space $\mathsf{S}$ *maximal*, if any Co-Polish space $\mathsf{X}$ embeds into $\mathsf{S}$ as a closed subspace. Our main result states that there are indeed maximal Co-Polish spaces.

**Theorem 2** *There exists a maximal Co-Polish space.*

It turns out that there is a plethora of non-homeomorphic maximal Co-Polish spaces. Indeed, if $S$ is a maximal Co-Polish space and $Y$ is any Co-Polish space, then both the product $S \times Y$ and the coproduct $S \oplus Y$ are maximal Co-Polish spaces. As a concrete example we consider the Hilbert space $\ell_2^{\gtrless}$ equipped with the sequentialisation of the weak*-topology, which is coarser than the usual norm topology on $\ell_2$. On the other hand, the Co-Polish space of polynomials is not maximal. Since forming closed subspaces preserves Co-Polishness, we obtain:

**Corollary 3** *A space is Co-Polish if, and only if, it is homeomorphic to a closed subspace of* $\ell_2^{\gtrless}$.

From Proposition 8 in [2] we conclude that overt choice on Co-Polish spaces has a maximum in the continuous Weihrauch lattice. Remember that overt choice on a represented space $X$ is the problem of finding a point in a non-empty closed subset of $X$ given by positive information.

**Corollary 4** *For any Co-Polish space* $\mathsf{X}$*, overt choice on* $\mathsf{X}$ *is continuously Weihrauch reducible to overt choice on* $\ell_2^{\geqslant}$*.*

### A tentative definition of a notion of a computable Co-Polish space

We now apply Corollary 3 to define a notion of a computable Co-Polish space. The Co-Polish Hilbert space $\ell_2^{\geqslant}$ is known to be the qcb-dual of the Polish Hilbert space $\ell_2$. So we can construct from the standard Cauchy representation for $\ell_2$ a natural effectively admissible representation for $\ell_2^{\geqslant}$, namely by co-restricting the function space representation for $\mathcal{C}(\ell_2, \mathbb{R})$ to the subspace of all linear continuous functions $f\colon \ell_2 \to \mathbb{R}$.

**Definition 5** A represented Co-Polish space $\mathsf{X}$ is *computable*, if there is an embedding $e\colon \mathsf{X} \hookrightarrow \ell_2^{\geqslant}$ such that $e$ and its inverse are computable and $e[X]$ is a co-recursively closed subset of $\ell_2^{\geqslant}$ with a computable dense subsequence.

In the talk I will discuss the properties of this notion. For example, computable Co-Polish spaces are closed under binary product. If $\mathsf{X}$ is a computable Co-Polish space then the lattice of opens $\mathcal{O}(\mathsf{X})$ equipped with the Scott topology is a computable quasi-Polish space in the sense of [2].

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# Formal proofs about metric spaces and continuity in coqrep

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Computable analysis makes continuous structures accessible to computation on digital computers by encoding their elements over Baire-space. Thus the central objects in computable analysis are representations: A representation for a space $X$ is a partial surjective mapping from the Baire space to $X$, i.e., some $\delta\colon\subseteq \mathbb{N}^{\mathbb{N}} \to X$ [KW85]. A pair $\mathbf{X} = (X, \delta_{\mathbf{X}})$ of a space and its representation is called a represented space and an element $\varphi$ of Baire space is called name of $x \in \mathbf{X}$ if $\delta(\varphi) = x$.

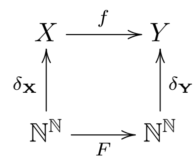


Figure 1: $F\colon\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is a realizer of $f\colon \mathbf{X} \to \mathbf{Y}$

The computability and topological structure of Baire space can be pushed forward through a representation. For instance one may say that a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of a represented space converges to another element $x$, i.e., $\lim x_n = x$ if there is a convergent sequence of names $(\varphi_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}^{\mathbb{N}}$ such that each $\varphi_n$ is a name for $x_n$ and the limit of the sequence of names is a name of $x$. A function between represented spaces is called sequentially continuous if it preserves this notion of a limit, i.e., if $\lim x_n = x$ implies that $\lim f(x_n) = f(x)$. A similar but in general slightly stronger notion of continuity is that of continuous realizability. A partial operator on Baire-space is said to realize a function $f : \mathbf{X} \to \mathbf{Y}$ between represented spaces if it takes names of $x$ to names of $f(x)$ (compare fig. 1). It is always true that a continuous realizable function is sequentially continuous, the opposite direction may fail in general but holds if the involved spaces are admissible [Sch03].

Many of the spaces that appear in applications come with additional structure. For instance, it is often the case that the space one wants to compute on comes with a metric. In this case there is notions of a metric limit and metric continuity is commonly defined by the $\varepsilon$-$\delta$ property. In this case one may want to pick a representation that reproduces these notions. Metric spaces have been considered in computable analysis, e.g., by Weihrauch [Wei93] and in particular in the case where $(M, d)$ is a metric space and $(r_n)_{n\in\mathbb{N}}$ is a designated dense

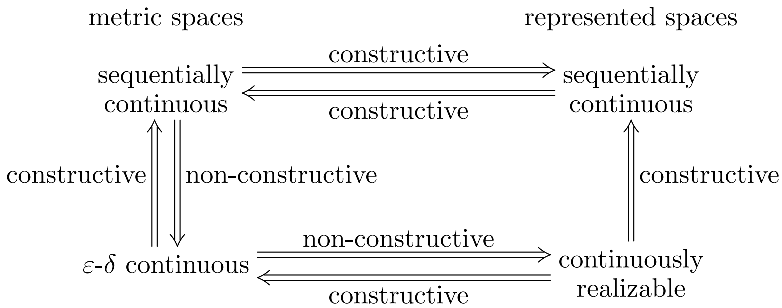


Figure 2: Implications between different notions of continuity on metric spaces

sequence, it is well known that

$$\delta_C(\varphi) = x \quad \iff \quad \forall n, d(x, r_{\varphi(n)}) \leq 2^{-n}.$$

defines a representation $\delta_C$ of $M$ that is known to be appropriate in the sense that the metric convergence relation coincides with the one introduced above.

We formalized the proofs of equivalences between the different notions of continuity using and extending the coqrep library for computable analysis in coq [Ste18]. We formally proved that sequential continuity on a metric space is equivalent to sequential continuity on the corresponding represented space, that $\varepsilon$-$\delta$-continuity is equivalent to the existence of a continuous realizer and that sequential continuity on a represented space implies the existence of a continuous realizer. A proof that $\varepsilon$-$\delta$-continuity implies sequential continuity can be found in the standard library such that we obtain all possible implications (compare fig. 2). The proofs have been kept as constructive as possible such that computational content can be extracted. Since the definition of a metric space relies on the axiomatic reals, only one of the implications is fully constructive, the others are constructive over the background theory of real numbers and do not rely on the axioms of the real numbers in an essential way. Some implications additionally need countable choice and functional extensionality.

While the results are fairly basic, most of the proofs are straightforward, and the use of the computational content is limited, they are important building stones for further work with metric spaces in the coqrep library. The original motivation for this work was to generalize previous work about isomorphy and discontinuity of spaces of subsets of represented spaces [ST19] from natural numbers to metric spaces. They also provide a basis for a possible future formalization of the computable Weierstraß approximation theorem [PEC75].

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