An update on Weihrauch reducibility (and some open questions)

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A brief history of Weihrauch degrees

- 1991 Weihrauch proposes and studies a precursor notion
- 2008 Gherardi and Marcone present the modern definition at CCA
- 2009 Brattka & Gherardi present the “manifesto” at CCA
- 2015 Dagstuhl meeting *Measuring the Complexity of Computational Content*
- 2018 Dagstuhl meeting *Measuring the Complexity of Computational Content: From Combinatorial Problems to Analysis*

2017: The survey

Vasco Brattka, Guido Gherardi & Arno Pauly:
Weihrauch Complexity in Computable Analysis.
arXiv 1707.03202

Goal today

What happened since? What are some interesting open questions?

Arno Pauly:
An update on Weihrauch complexity, and some open questions.
arXiv 2008.11168

A very short overview

- Weihrauch reducibility compares multivalued functions between represented spaces.
- The induced degrees have a rich algebraic structure.
- Many mathematical theorems can be interpreted as multivalued functions, with the associated Weihrauch degrees measuring the computational content of the theorem.
- The algebraic operations have logic-like meanings regarding such theorems.
- Many concrete theorems have been classified via Weihrauch reducibility; and this classification is reminiscent of reverse mathematics and Brouwerian counterexamples.
- Various techniques have been developed to prove separation results.

Weihrauch-reducibility

Definition

For $f :\subseteq \mathbf{X} \rightrightarrows \mathbf{Y}$, $g :\subseteq \mathbf{V} \rightrightarrows \mathbf{W}$ say

$$f \leq_W g$$

iff there are computable $H, K :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$, such that $H\langle \mathrm{id}_{\mathbb{N}^{\mathbb{N}}}, GK\rangle$ is a realizer of f for every realizer G of g.

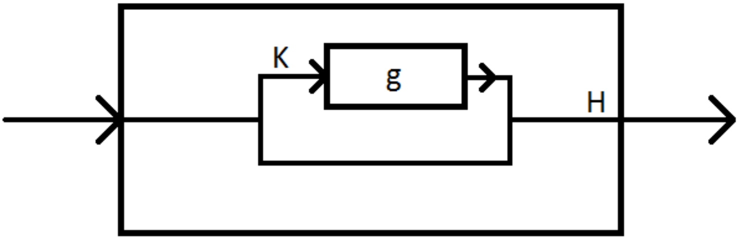

Figure: Weihrauch reducibility

Outline

Algebraic structure

We have the following operations on Weihrauch degrees:

1. $f \sqcap g$, returning either an answer to f or an answer to g
2. $f \sqcup g$, letting us choose between f and g
3. $f \times g$, letting us both f and g in parallel
4. $f \star g$, letting us first use g, then f
5. $f \to g = \min\{h \mid g \leq_W f \star h\}$
6. $f^*, f^\diamond$ letting us use f finitely many times, in parallel or consecutively
7. $\widehat{f}$, letting us use f countably many times in parallel
8. (and more)

Algebraic structure, continued

Theorem

- $(\mathfrak{W}, \sqcap, \sqcup)$ *is a distributive lattice.*
- *It lacks some infinite infima, and all non-trivial infinite suprema.*
- *It is not a Heyting algebra.*
- $(\mathfrak{W}, \sqcup, \times, ^*)$ *is a Kleene-algebra.*

Characterizations

Proposition

f^* is the least Weihrauch degree above f satisfying $1 \leq_W f^*$ and $f^* \times f^* \equiv_W f^*$.

Theorem (Westrick 2020)

$f^\diamond$ is the least Weihrauch degree above f satisfying $1 \leq_W f^\diamond$ and $f^\diamond \star f^\diamond \equiv_W f^\diamond$.

- Open since CCA 2015
- There is a constant function f and a multivalued function g such that $f \leq_W g^\diamond$, but no fixed finite number of applications of g suffices

More definability?

- Clearly $\sqcup$, $\sqcap$, $\emptyset$, 0 are definable just by $\leq_{\mathrm{W}}$
- Are $\times$ or 1 definable from other operations? What about $\widehat{\ }$?

On the theory of Weihrauch degrees

- The Weihrauch degrees are a distributive lattice.
- Every countable distributive lattice embeds into the Weihrauch degrees.
- Thus, any universally quanitified statement using $\sqcup$ and $\sqcap$ is either provable from the axioms of distributive lattices or false in $\mathfrak{W}$.
- Can we extend this to additional operations?

The scaffolding

Traditionally, we compare Weihrauch degrees of theorems to

- Closed choice principles
- Principles such as LPO reflecting Brouwerian counterexamples
- The effective Baire hierarchy via iterations of $\lim$

What happens if we explore away from that familiar area?

Overt choice

Definition (Overt Choice)

Let $\mathrm{VC}_{\mathbf{X}} :\subseteq \mathcal{V}(\mathbf{X}) \rightrightarrows \mathbf{X}$ map $A \in \mathcal{V}(\mathbf{X})$ to some $x \in A$.

Theorem (de Brecht, P. & Schröder)

1. *If $f : \mathbf{X} \rightrightarrows \mathbb{N}$ is non-computable, then $f \nleq_W \mathrm{VC}_{\mathbb{Q}}$.*
2. $\mathrm{NON} \nleq_W \mathrm{VC}_{\mathbb{Q}}$
3. $\mathrm{VC}_{\mathbb{Q}} \nleq_W \mathrm{C}_{\mathbb{N}}$

The deterministic part

Definition (Goh, P. & Valenti)

$$\mathrm{Det}_{\mathbf{X}}(f) := \max_{\leq_{\mathrm{W}}}\{F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbf{X} \mid F \leq_{\mathrm{W}} f\}$$

Abbreviate $\mathrm{Det}(f) = \mathrm{Det}_{\mathbb{N}^{\mathbb{N}}}(f)$

Different codomains often yield different operators $\mathrm{Det}_{\mathbf{X}}$, as shown via the point spectrum of the space.

Finding descending sequences

Let DS be "find a descending sequence in a non-wellfounded linear order".

Theorem (Goh, P. & Valenti)

$\mathrm{DS} \star \mathrm{DS} \equiv_W \mathrm{C}_{\mathbb{N}^\mathbb{N}}$, *but* $\mathrm{Det}(\mathrm{DS}) \equiv_W \lim$.

Question

Does $\mathrm{C}'_{\mathbf{2}^\mathbb{N}} \leq_W \mathrm{DS}$ *hold?*

Listing elements of a set

Definition

$\mathrm{List}_{\leq\omega} :\subseteq \mathcal{A}(\mathbf{2}^{\mathbb{N}}) \rightrightarrows \mathbf{2}^{\mathbb{N}}$ maps countable closed subsets of $\mathbf{2}^{\mathbb{N}}$ to enumerations of their elements.

Theorem (Kihara, Marcone & P.)

$\mathrm{List}_{\leq\omega} \star \mathrm{List}_{\leq\omega} \star \mathrm{List}_{\leq\omega} \equiv_W \mathrm{UC}_{\mathbb{N}^{\mathbb{N}}}$, *but* $\mathrm{Det}(\mathrm{List}_{\leq\omega}) \equiv_W \lim$.

Question

$\mathrm{List}_{\leq\omega} \star \mathrm{List}_{\leq\omega} \equiv_W \mathrm{UC}_{\mathbb{N}^{\mathbb{N}}}$ *hold?*

Question

Does $\mathrm{List}_{\leq\omega} \leq_W \mathrm{DS}$ *hold?*

Ramsey's theorem

Theorem (Dzhafarov, Goh, Hirschfeldt, Patey & P.)

$\mathrm{NON} \times LPO \nleq_W \mathrm{RT}^2_2$

Question

Does $\mathrm{NON} \times LLPO \leq_W \mathrm{RT}^2_2$ *hold?*

All-or-unique choice

Definition

Let $\mathrm{AoUC} :\subseteq \mathcal{A}(\mathbf{2}^{\mathbb{N}}) \rightrightarrows \mathbf{2}^{\mathbb{N}}$ be the restriction of $\mathrm{C}_{\mathbf{2}^{\mathbb{N}}}$ to $\{\mathbf{2}^{\mathbb{N}}\} \cup \{\{p\} \mid p \in \mathbf{2}^{\mathbb{N}}\}$.

Proposition (Brattka)

AoUC *is equivalent to "Solve* $bx = a$ *for* $0 \leq a \leq b$*".*

Theorem (P.; Kihara & P.; Crook & P.)

AoUC^* *is equivalent to finding Nash equilibria in bimatrix games; to a version of Gaussian elimination and to finding roots of finitely many univariate polynomials.*

A hickup

Theorem (Kihara & P.)

$\mathrm{AoUC}^* <_W \mathrm{AoUC}^* \star \mathrm{AoUC}^* \equiv_W \mathrm{AoUC}^\diamond$

Question

Is there any "natural" problem in the degree $\mathrm{AoUC}^\diamond$*?*

A question on the Wadge game

Question (Nobrega &P.)

What is the Weihrauch degree of obtaining a point of discontinuity of a function from a Player I winning strategy in the Wadge game for it?

Some classifications

- Fiori-Carones, Shafer and Soldà studied the Rival-Sands theorem, and proved it to be equivalent to WKL''.
- Kohlenbach has studied the Weihrauch degree of the modulus of regularity.
- Finding a projection of point to a closed (or overt) subset of $\mathbb{R}^n$ was classified by Gherardi, Marcone and P.

And more

- Steinberg, Théry and Thies demonstrated the feasibility of doing computable analysis in Coq.
- Uftring obtained a proof-theoretic characterisation of Weihrauch reducibility.
- Day, Downey and Westrick showed that Bourgain's α-rank can be characterized via Weihrauch reducibility.