Complexity and Coding Theory of Hilbert Spaces[1]

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Complexity and Coding Theory of Hilbert Spaces

Fact

Every infinite-dimensional separable Hilbert spaces are isomorphic.

Example

$\mathcal{L}^2$ is the space of square-integrable functions $f : [0,1] \to \mathbb{C}$ equipped with the inner product

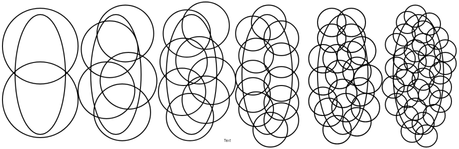
$$\langle f, g \rangle := \int_0^1 f(x)\overline{g(x)}dx.$$

Example

$\{e^{2\pi kix}\}_{k\in\mathbb{Z}}$ is an orthonormal basis of $\mathcal{L}^2$.

Definition

$\eta : \mathbb{N} \to \mathbb{N}$ is the **entropy** of space X if for every $n \in \mathbb{N}$, X can be covered by $2^{\eta(n)}$ closed balls of radius 2^{-n}, but not by $2^{\eta(n)-1}$ closed balls.



Example

The unit interval $[0, 1]$ has the entropy $\eta(n) = n$.

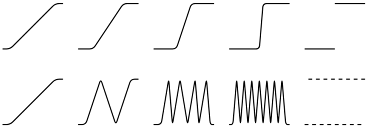
Example (Steinberg, 2017)

$\mathrm{Lip}_1([0, 1], [0, 1])$ has the entropy $\eta(n) = 2^{\mathcal{O}(n)}$.

Definition

$\mu : \mathbb{N} \to \mathbb{N}$ is a **modulus of continuity** of $f : (X, d) \to (Y, e)$ if

$$\forall n \in \mathbb{N} \quad \forall a, b \in X \quad d(a, b) \leq 2^{-\mu(n)} \text{ implies } e(f(a), f(b)) \leq 2^{-n}$$



Dyadic Representation: $\{0, 1\}^{\omega} \twoheadrightarrow [0, 1]$

$$bin(a_0)bin(a_1)bin(a_2)\cdots \mapsto \lim_{n\to\infty} \frac{a_n}{2^n}$$

Signed Binary Representation: $\{0, 1\}^{\omega} \twoheadrightarrow [0, 1]$

$$bin(a_0)bin(a_1)bin(a_2)\cdots \mapsto \sum_{n=0}^{\infty} \frac{a_n}{2^n}$$

$a_i \in \{-1, 0, 1\}$

Dyadic Representation: $\{0,1\}^\omega \twoheadrightarrow [0,1]$; modulus $\Theta(n^2)$

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Signed Binary Representation: $\{0,1\}^\omega \twoheadrightarrow [0,1]$; modulus $\Theta(n)$

$$bin(a_0)bin(a_1)bin(a_2)\cdots \mapsto \sum_{n=0}^{\infty} \frac{a_n}{2^n}$$

$a_i \in \{-1,0,1\}$

Steinberg's Lemma (2016)

For a surjection $f : X \to Y$ and its modulus μ,

$$\forall n \quad \eta_X(n) \leq \eta_Y(\mu(n))$$

where η_X and η_Y being the entropy of X and Y, respectively.

- Sometimes we want to find f with (asympotitically) small μ.
- This lemma establishes a lower bound of μ.

Weihrauch-style Representation

A partial surjection $\{0,1\}^\omega \twoheadrightarrow X$

Kawamura-style Representation (2012)

A partial surjection $(\{0,1\}^* \to \{0,1\}^*) \twoheadrightarrow X$

Our-style Representation

A partial surjection $(\{0,1\}^* \to \{0,1\}) \twoheadrightarrow X$ with metric d on domain

$$d(\psi, \varphi) := 2^{-\min\{|w| : \psi(w) \neq \varphi(w)\}}$$

- ψ and φ are close iff they agree on all $w \in \{0,1\}^*$ up to some length.

Standard Representation of $\mathrm{Lip}_1([0,1],[0,1])$

$$(\{0,1\}^* \to \{0,1\}) \twoheadrightarrow \mathrm{Lip}_1([0,1],[0,1])$$

$\varphi \mapsto f$ iff $\varphi(0^n 1^j 0 bin(A)) = j$th bit in the binary encoding of $f(A/2^n)$ approximated to precision 2^{-n}

- $\mathrm{Lip}_1([0,1],[0,1])$ is compact according to Arzela-Ascoli
- The standard representation has a linear modulus.
- n and j are encoded in unary and A is encoded in binary.
- This modulus is (asymptotically) optimal by Steinberg's lemma since both $(\{0,1\}^* \to \{0,1\})$ and $\mathrm{Lip}_1([0,1],[0,1])$ have exponential entropy.

Standard Representation of $\mathrm{Lip}_1([0,1],[0,1])$

$$(\{0,1\}^* \to \{0,1\}) \twoheadrightarrow \mathrm{Lip}_1([0,1],[0,1])$$

$\varphi \mapsto f$ iff $\varphi(0^n 1^j 0 bin(A)) = j$th bit in the binary encoding of $f(A/2^n)$ approximated to precision 2^{-n}

- Use of $(\{0,1\}^* \to \{0,1\})$ instead of $\{0,1\}^* \to \{0,1\}^*$ eliminates the need for second-order complexity.
- Application functional $(f,x) \mapsto f(x)$ is polytime computable with respect to standard representation. (Kawamura, 2012)
- A representation δ of $\mathrm{Lip}_1([0,1],[0,1])$ is polytime reducible to standard representation iff it makes application functional polytime computable. (Kawamura, 2012)

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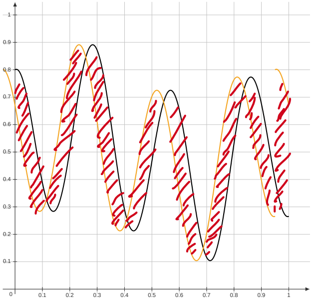
Definition

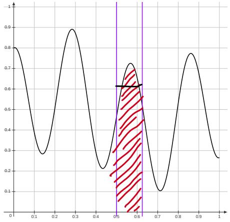
$\mathcal{L}^2_1$ is the space of functions $f : [0,1] \to \mathbb{C}$ such that

$$\|f\| \leq 1 \quad \wedge \quad \forall \epsilon > 0 \; \|\tau_\epsilon f - f\| \leq \epsilon$$

where $\|f\|$ is the norm $\|f\| := \sqrt{\int_0^1 f(x)\overline{f(x)}dx}$
and τ is the cyclic shift $(\tau_\epsilon f)(x) := f(x + \epsilon \bmod 1)$.



- $\mathcal{L}^2_1$ is compact according to Fréchet-Kolmogorov

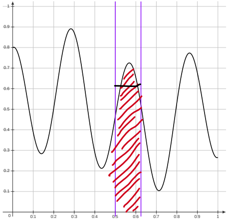


Piecewise Representation of $\mathcal{L}_1^2$

$$(\{0,1\}^* \to \{0,1\}) \twoheadrightarrow \mathcal{L}_1^2$$

$\varphi \mapsto f$ iff $\varphi(0^n 1^j 0 bin(A)) = j$th bit in the binary encoding of $2^n \cdot \int_{A/2^n}^{(A+1)/2^n} f$ approximated up to precision 2^{-n}

- Similar representations have been used for computability investigations.
- $\mathcal{L}_1^2$ has entropy $\eta(n) = 2^{\mathcal{O}(n)}$ (Steinberg, 2016)
- $(\{0,1\}^* \to \{0,1\})$ has entropy $\eta(n) = 2^{\mathcal{O}(n)}$.
- Piecewise representation has linear modulus and this is optimal.



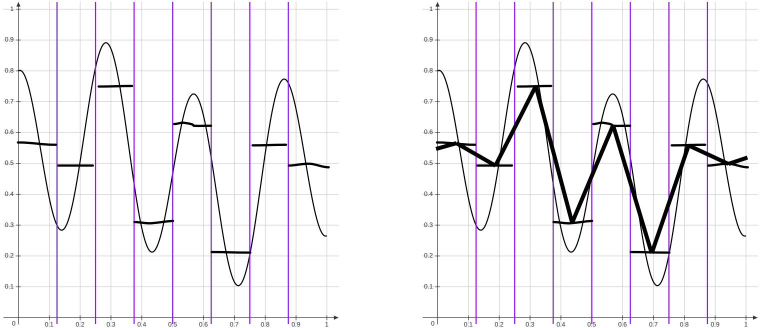
Theorem

Under piecewise representation, the functional $f \mapsto \int_0^1 f$ is not polytime but exptime computable.

Proof. Because exponentially many intervals cannot be accessed in polytime. (Made formal by perturbation argument.)

Conjecture

Under piecewise representation, the pointwise operator $(f, g) \mapsto \sqrt{f \cdot g}$ is polytime computable.



- Let $f \in \mathcal{L}^2_1$.
- For each interval, we can calculate the average by integration.
- Not closed under piecewise constant approximation $f \in \mathcal{L}^2_1 \not\Rightarrow \tilde{f} \in \mathcal{L}^2_1$.
- Closed under piecewise *linear* approximation $f \in \mathcal{L}^2_1 \Rightarrow \hat{f} \in \mathcal{L}^2_1$.

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Definition

ℓ^2_1 is the space of sequences $(z_K)_{K\in\mathbb{Z}} \subseteq \mathbb{C}$ such that

$$\sqrt{\sum_K |z_K|^2} \leq 1 \quad \wedge \quad \sqrt{\sum_K |2\pi i K z_K|^2} \leq 1.$$

Lemma

ℓ^2_1 is isometric to $\mathcal{L}^2_1$.
Proof. By Parseval's identity and some additional arguments.

Fourier Coefficient Representation

$$(\{0,1\}^* \to \{0,1\}) \twoheadrightarrow \ell^2_1$$

$\varphi \mapsto (z_K)_{K\in\mathbb{Z}}$ iff $\varphi(0^n 1^j 0 bin(K)) = j$th bit in the binary encoding of z_K approximated up to precision 2^{-n}

- Since ℓ^2_1 can be isometrically identified with $\mathcal{L}^2_1$, a representation of ℓ^2_1 is also a representation of $\mathcal{L}^2_1$ and vice versa.

Fourier Coefficient Representation

$$(\{0,1\}^* \to \{0,1\}) \twoheadrightarrow \ell_1^2$$

$\varphi \mapsto (z_K)_{K\in\mathbb{Z}}$ iff $\varphi(0^n 1^j 0 bin(K)) =$ jth bit in the binary encoding of z_K approximated up to precision 2^{-n}

- n and j encoded in unaryy; K encoded in binary
- Fourier coefficient representation has linear modulus and this is optimal by Steinberg's lemma.

Observation

In Fourier representation, the functional $f \mapsto \int_0^1 f$ is polytime computable by directly reading it off from the encoding.

A Characterization of Fourier Coefficient Representation

A representation δ is polytime reducible to Fourier coefficient representation iff δ makes polytime computable the functional $(f, \mathrm{bin}(K)) \mapsto K$-th Fourier coefficient.

Fourier Coefficient Representation

$$(\{0,1\}^* \to \{0,1\}) \twoheadrightarrow \ell_1^2$$

$\varphi \mapsto (z_K)_{K\in\mathbb{Z}}$ iff $\varphi(0^n 1^j 0 bin(K)) = j$th bit in the binary encoding of z_K approximated up to precision 2^{-n}

Conjecture

The functional $f \mapsto \int_0^{1/3} f$ is not polytime but exptime computable under Fourier coefficient representation.

Conjecture

The functional $(f, g) \mapsto \int_0^1 f(x)\overline{g(x)}dx$, the 0th Fourier coefficient of $f \cdot g$, is not polytime computable but exptime computable under Fourier coefficient representation. (Can be computed by convolution)

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Piecewise Representation of $\mathcal{L}^2_1$

$$(\{0,1\}^* \to \{0,1\}) \twoheadrightarrow \mathcal{L}^2_1$$

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Fourier Coefficient Representation

$$(\{0,1\}^* \to \{0,1\}) \twoheadrightarrow \ell^2_1$$

$\varphi \mapsto (z_K)_{K\in\mathbb{Z}}$ iff $\varphi(0^n 1^j 0 bin(K)) = j$th bit in the binary encoding of z_K approximated up to precision 2^{-n}

Conjecture

	Piecewise Repr	Fourier Coef Repr
$f \mapsto \int_0^1 f$	Not polytime	Polytime
$(f, unary(n)) \mapsto 2^n \cdot \int_0^{2^{-n}} f$	Polytime	Not polytime

- Therefore the two representations are not polytime convertible.

Summary

- $\mathcal{L}_1^2$ and piecewise representation.
- ℓ_1^2 and Fourier Coefficient representation.
- The two are not polytime equivalent.
- One needs to pick one depending on the purpose.

Future Work

- With respect to the basis $\{e^{2\pi kix}\}_{k\in\mathbb{Z}}$ of $\mathcal{L}_1^2$, we get Fourier coefficients.
- What if we use another orthonormal basis?
- Is there a functional that characterizes the piecewise representation?
- Combining representations makes more functionals polytime computable, but may destroy closure under operations.