Isometries and the equivalence of the effective separating sequences

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Computability on a metric space

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- $\beta : \mathbb{N} \to \mathbb{N}$ is **a computable sequence** in (X, d, α) if there exists a computable function $F : \mathbb{N}^2 \to \mathbb{N}$ such that $d(\beta_i, \alpha_{F(i,k)}) < 2^{-k}$, for all $i, k \in \mathbb{N}$.

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Are those notions defined by the metric space itself?

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Under which circumstances are all effective separating sequences equivalent?

Two examples

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$(\boldsymbol{[0,1]}, \boldsymbol{d}, \boldsymbol{q})$

- if β is an effective separating sequence, 0 is computable in $([0,1], d, \beta)$
- $i \mapsto d(0, \beta_i)$ is computable, so $\beta : \mathbb{N} \to \mathbb{R}$ is computable
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(S^1, d, α)

- if x is a computable point and y a non-computable point, there exists a rotation f such that $f(x) = y$
- $f \circ \alpha$ is an effective separating sequence and y is computable in $(S^1, d, f \circ \alpha)$
- $f \circ \alpha \not\sim \alpha$

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(X, d, α) is **an effectively (or computably) compact computable metric space** if (X, d) is complete and there exists a computable function $f : \mathbb{N} \to \mathbb{N}$ such that $X = \bigcup_{i=0}^{f(k)} B(\alpha_i, 2^{-k})$, for each $k \in \mathbb{N}$.

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Theorem 1 (Iljazović, 2010)

Let (X, d, α) be an effectively compact computable metric space such that there exist only finitely many isometries of the metric space (X, d). If β is an effective separating sequence in (X, d), then $\beta \sim \alpha$.

New result

Theorem 2

Let (X, d, α) be an effectively compact metric space and K a computable compact set in (X, d, α) such that there are only finitely many isometries $f : X \to X$ such that $f(K) \subseteq K$. If β is an effective separating sequence in (X, d) such that K is computable in (X, d, β), then $\alpha \sim \beta$.

Sketch of the proof

Idea: Define a metric space (Y, D) whose isometries are in a 1-1 correspondence with isometries of (X, d) such that $f(K) \subseteq K$.

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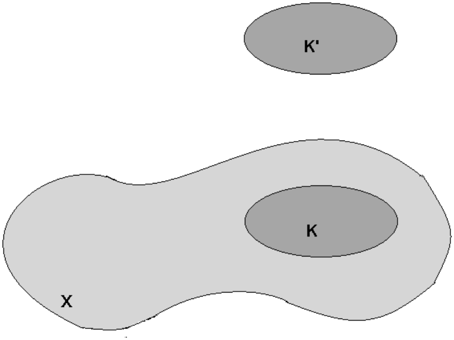

$$K' = K \times \{\gamma\},\ Y = X \cup K'$$

Take $M \in \mathbb{N}$ such that $M > \text{diam}(X)$. Define $D : Y \times Y \to \mathbb{R}$,

$$D(x, y) = d(x, y),\ x, y \in X$$
$$D((k, \gamma), x) = D(x, (k, \gamma)) = M + d(x, k),\ x \in X, k \in K$$
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Using sequences α and $((x_i, \gamma))$ we can define an effective separating sequence δ in (Y, D) such that (Y, D, δ) is an effectively compact computable metric space.

Consequence:

Proposition 3

Assume that (X, d, α) is an effectively compact computable metric space and $x_0, \dots, x_n$ computable points in (X, d, α) such that there are only finitely many isometries $f : X \to X$ such that $f(x_i) = x_i$, $i = 0, \dots, n$. If β is an effective separating sequence such that $x_0, \dots, x_n$ are computable points in (X, d, β), then $\alpha \sim \beta$.

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$([0, \gamma], d, \alpha)$, γ **left computable, not computable**

- β an effective separating sequence such that $\frac{\gamma}{2}$ is computable in $([0, \gamma], d, \beta)$
- $\frac{\gamma}{2}$ is a fixed point of each isometry and is not computable in $([0, \gamma], d, \alpha)$
- $\alpha \not\sim_{\text{iso}} \beta$

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Proposition 3 $\Rightarrow \;\beta \sim f \circ \alpha \;\Rightarrow\; \beta \sim_{\text{iso}} \alpha$

Generalisation in $\mathbb{R}^2$

Theorem 4

Assume that X is a subset of $\mathbb{R}^2$, d is the Euclidean metric on X and that there are infinitely many isometries of the metric space (X, d). If α is a sequence such that (X, d, α) is an effectively compact metric space and β is an effective separating sequence in (X, d), then α and β are equivalent up to an isometry.

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ΔX contains computable points in both computable metric spaces, and we can apply a rotation in the same way as in S^1.

Conclusion for $\mathbb{R}^2$

If a set in $\mathbb{R}^2$ admits a structure of an effectively compact computable metric space, then any two effective separating sequences in that set are equivalent up to an isometry!

Further generalisations?