Weihrauch and constructive reducibility between existence statements

Makoto Fujiwara

JSPS Research Fellow (PD), Meiji University/LMU Munich

CCA 2020 (virtual)
10 September 2020

This work is supported by JSPS KAKENHI Grant Numbers 18K13450 and 19J01239, as well as JSPS Core-to-Core Program (A. Advanced Research Networks).

Introduction

- Many existence statements in mathematics can be formalized as Π_2 sentences of form

$$\forall f \, (A(f) \to \exists g B(f, g)) \, .$$

- In general, f and g are possibly tuples of functions respectively, but in this talk, we present Π_2 sentences as above for notational simplicity. Here f is called an **instance** and g is called a **solution** to f.
- During these decades, the interrelations between existence statements have been studied extensively in several contexts of reverse mathematics (RM).

There are several corresponding results between the following two RM:

1. RM via Weihrauch reducibility from Computable Analysis;
2. RM over Constructive Mathematics.

There are several corresponding results between the following two RM:

1. RM via Weihrauch reducibility from Computable Analysis;
2. RM over Constructive Mathematics.

Fact. (Weihrauch RM)

$\mathrm{DICH}_{\mathbb{R}}$ is Weihrauch equivalent to LLPO.

Fact. (Const. RM over BISH)

$\mathrm{DICH}_{\mathbb{R}}$ is constructively equivalent to LLPO.

- $\mathrm{DICH}_{\mathbb{R}} : \forall \alpha \in \mathbb{R} \, (\alpha \geq 0 \vee \alpha \leq 0)$.
- LLPO : $\forall f^{\mathbb{N}\to\mathbb{N}} \left(\begin{array}{l} \neg \left(\exists n^{\mathbb{N}} f(2n) = 0 \wedge \exists n^{\mathbb{N}} f(2n+1) = 0\right) \\ \to \forall n^{\mathbb{N}} f(2n) \neq 0 \vee \forall n^{\mathbb{N}} f(2n+1) \neq 0 \end{array} \right)$

Definition.

The **parallelization** (or sequential version) $\widehat{\mathrm{P}}$ of $\mathrm{P} \equiv \forall f(A(f) \to \exists g B(f,g))$ is

$$\forall \langle f_n \rangle_{n\in\mathbb{N}} \left(\forall n\, A(f_n) \to \exists \langle g_n \rangle_{n\in\mathbb{N}} \forall n\, B(f_n, g_n)\right).$$

Remark. $\mathrm{P} \leq_W \mathrm{Q} \Longrightarrow \mathrm{P} \leq_W \widehat{\mathrm{Q}}$.

Definition.

The **parallelization** (or sequential version) $\widehat{\mathrm{P}}$ of $\mathrm{P} \equiv \forall f(A(f) \to \exists g B(f,g))$ is

$$\forall \langle f_n \rangle_{n\in\mathbb{N}} \left(\forall n\, A(f_n) \to \exists \langle g_n \rangle_{n\in\mathbb{N}} \forall n\, B(f_n, g_n)\right).$$

Remark. $\mathrm{P} \leq_W \mathrm{Q} \Longrightarrow \mathrm{P} \leq_W \widehat{\mathrm{Q}}$.

Fact. (Weihrauch RM)

1. WKL is Weihrauch reducible to $\widehat{\mathrm{IVT}}$ (and vice versa) but not so to IVT.
2. WKL is Weihrauch reducible to $\widehat{\mathrm{LLPO}}$ (and vice versa) but not so to LLPO.

Definition.

The **parallelization** (or sequential version) $\widehat{\mathrm{P}}$ of $\mathrm{P} \equiv \forall f (A(f) \to \exists g B(f, g))$ is

$$\forall \langle f_n \rangle_{n\in\mathbb{N}} (\forall n\, A(f_n) \to \exists \langle g_n \rangle_{n\in\mathbb{N}} \forall n\, B(f_n, g_n))\,.$$

Remark. $\mathrm{P} \leq_W \mathrm{Q} \Longrightarrow \mathrm{P} \leq_W \widehat{\mathrm{Q}}$.

Fact. (Weihrauch RM)

1. WKL is Weihrauch reducible to $\widehat{\mathrm{IVT}}$ (and vice versa) but not so to IVT.
2. WKL is Weihrauch reducible to $\widehat{\mathrm{LLPO}}$ (and vice versa) but not so to LLPO.

Fact. (Const. RM over BISH which accepts $\mathrm{AC}^{0,\omega}$)

$\mathrm{IVT} \Leftrightarrow \mathrm{WKL} \Leftrightarrow \mathrm{LLPO}$.

- In this decade, there are several attempts to characterize Weihrauch RM from such a formalistic approach (Kuyper 2017, Hirst-Mummert 2019, Uftring 2020 etc.).
- Here we partially characterize the notions of $\mathrm{P} \leq_W \mathrm{Q}$ and $\mathrm{P} \leq_W \widehat{\mathrm{Q}}$ in Weihrauch RM by some derivability notions observed in Constructive RM.
- In these two decades, Constructive RM, as well as some of Weihrauch RM, have been developed over (many-sorted) arithmetic as for Friedman-Simpson RM.
- Our approach:

Weihrauch RM		Const. RM over BISH
$\wr$		$\wr$
Classical RM (Formal)	$\Longleftrightarrow$	Const. RM (Formal)

- We employ finite-type arithmetic as our framework.

- Hilbert-type system E-HA$^\omega$ (resp. E-PA$^\omega$) is the finite type extension of HA (resp. PA), of which **T** is the terms.
- $\widehat{\text{E-HA}}^\omega\restriction$ (resp. $\widehat{\text{E-PA}}^\omega\restriction$) is the restrictions of E-HA$^\omega$ (resp. E-PA$^\omega$) to **primitive recursion of type 0** and quantifier-free induction, of which $\mathbf{T_0}$ is the terms.
- Type-1 functions (functions of type $\mathbb{N}^\mathbb{N}$) definable in $\mathbf{T_0}$ (resp. **T**) coincide with primitive (resp. PA-provably) recursive functions in the ordinary sense.

$\mathbb{N}$		HA
$\mathbb{N}^\mathbb{N}$	EL_0	EL
ω	$\widehat{\text{E-HA}}^\omega\restriction + \text{QF-AC}^{0,0}$	$\text{E-HA}^\omega + \text{QF-AC}^{0,0}$

Fact. (Kohlenbach 2005)

$\text{RCA}_0^\omega :\equiv \widehat{\text{E-PA}}^\omega\restriction + \text{QF-AC}^{1,0}$ is a conservative extension of RCA_0 in Friedman-Simpson RM.

$$\text{AC}^{\sigma,\tau} : \forall x^\sigma \exists y^\tau A(x,y) \to \exists Y^{\tau^\sigma} \forall x^\sigma A(x, Yx).$$

Definition (Weihrauch reducibility for Π^1_2 statements)

For Π^1_2 statements P and Q of form $\forall f\,(A(f) \to \exists g B(f,g))$, P is **Weihrauch reducible** to Q (denoted as $\mathrm{P} \leq_W \mathrm{Q}$) if there exist Turing functionals Φ and Ψ such that whenever f is an instance of P, then $f' := \Phi(f)$ is an instance of Q, and whenever g' is a solution to f', then $g := \Psi(f \oplus g')$ is a solution to f.

Definition (Weihrauch reducibility for Π^1_2 statements)

For Π^1_2 statements P and Q of form $\forall f\, (A(f) \to \exists g B(f, g))$, P is **Weihrauch reducible** to Q (denoted as $\mathrm{P} \leq_W \mathrm{Q}$) if there exist Turing functionals Φ and Ψ such that whenever f is an instance of P, then $f' := \Phi(f)$ is an instance of Q, and whenever g' is a solution to f', then $g := \Psi(f \oplus g')$ is a solution to f.

- In the following, we define the **primitive recursive** (in the sense of Gödel/Kleene) variants of Weihrauch reducibility in which Turing functionals for the reduction are replaced by primitive recursive (total) functionals (in the sense of Gödel/Kleene).
- The verification theory is also concerned.

PR Variants of Weihrauch Reducibility in S^ω

Definition

Let $\mathrm{P} : \forall f\,(A_1(f) \to \exists g\, B_1(f,g))\,,\ \mathrm{Q} : \forall f\,(A_2(f) \to \exists g B_2(f,g))$.

- P **is Gödel-primitive-recursive Weihrauch reducible to** Q **in** S^ω if there exist closed terms s and t (of suitable types) in $\mathbf{T}$ such that S^ω proves

$$\forall f(A_1(f) \to A_2(sf)) \wedge \forall f, g' \left(B_2(sf, g') \wedge A_1(f) \to B_1(f, tfg')\right).$$

- P **is Kleene-primitive-recursive Weihrauch reducible to** Q **in** S^ω if there exist closed terms s and t (of suitable types) in $\mathbf{T_0}$ such that S^ω proves the same sentence.

Proposition. (cf. Brattka/Gherardi 2011)

WKL is Kleene-primitive-recursive Weihrauch reducible to $\widehat{\mathrm{LLPO}}$ in $\widehat{\textsf{E-PA}}^{\omega}\!\restriction + \mathrm{QF\text{-}AC}^{0,0}$ (which contains Π^0_1-IND).

Remark.

$\widehat{\textsf{E-PA}}^{\omega}\!\restriction + \mathrm{QF\text{-}AC}^{0,0}$ and $\widehat{\textsf{E-PA}}^{\omega}\!\restriction + \Pi^0_1\text{-}\mathrm{AC}^{0,0}$ are conservative extensions of RCA_0 and ACA_0 in Friedman-Simpson RM respectively.

Definition (Normal Reducibility in S^ω)

Let $\mathrm{P} : \forall f \left(A_1(f) \to \exists g\, B_1(f,g)\right)$, $\mathrm{Q} : \forall f \left(A_2(f) \to \exists g B_2(f,g)\right)$.
We say that P **is normally reducible to** Q **in** S^ω if S^ω proves

$$\forall f \left(A_1(f) \to \exists f' \left(A_2(f') \wedge \forall g' \left(B_2(f',g') \to \exists g B_1(f,g)\right)\right)\right).$$

- The normal reducibility, which requires a specific form of a proof of that Q implies P, is a stronger notion than just proving $\mathrm{Q} \to \mathrm{P}$.
- Since intuitionistic finite-type arithmetic with a choice principle roughly corresponds to Bishop's constructive mathematics, one may regard the normal reducibility in a nearly intuitionistic finite-type arithmetic as a sort of constructive reducibility.

The normal reducibility in the context of a classical system is nothing but provability in the system:

Proposition.

Let $\mathrm{P} : \forall f \, (A_1(f) \to \exists g \, B_1(f, g))$, $\mathrm{Q} : \forall f \, (A_2(f) \to \exists g B_2(f, g))$, and S^ω be a classical finite-type arithmetic such that $\mathsf{S}^\omega \vdash \exists f' A_2(f')$.*
If $\mathsf{S}^\omega \vdash \mathrm{Q} \to \mathrm{P}$, then P is normally reducible to Q in S^ω.

*Note that if $\mathsf{S}^\omega \vdash \forall f' \neg A_2(f')$, then $\mathsf{S}^\omega \vdash \mathrm{Q} \to \mathrm{P}$ just means $\mathsf{S}^\omega \vdash \mathrm{P}$

The normal reducibility in the context of a classical system is nothing but provability in the system:

Proposition.

Let $\mathrm{P} : \forall f\,(A_1(f) \to \exists g\, B_1(f,g))$, $\mathrm{Q} : \forall f\,(A_2(f) \to \exists g B_2(f,g))$, and S^ω be a classical finite-type arithmetic such that $\mathsf{S}^\omega \vdash \exists f' A_2(f')$.*
If $\mathsf{S}^\omega \vdash \mathrm{Q} \to \mathrm{P}$, then P is normally reducible to Q in S^ω.

Remark. The above proposition does not hold for intuitionistic finite-type arithmetic. Thus, in an intuitionistic context, the notion of normal reducibility is a strictly stronger notion than provability for existence statements.

*Note that if $\mathsf{S}^\omega \vdash \forall f' \neg A_2(f')$, then $\mathsf{S}^\omega \vdash \mathrm{Q} \to \mathrm{P}$ just means $\mathsf{S}^\omega \vdash \mathrm{P}$

Characterization of a Weakening of $\mathrm{P} \leq_W \mathrm{Q}$

Proposition.

Let $\mathrm{P} : \forall f\,(A_1(f) \to \exists g\, B_1(f,g))$, $\mathrm{Q} : \forall f\,(A_2(f) \to \exists g B_2(f,g))$ with $\exists$-free (containing neither $\exists$ nor $\vee$) formulas A_1, A_2, B_1, B_2.

- P is Gödel-primitive-recursive Weihrauch reducible to Q in $\text{E-PA}^\omega \iff \mathrm{P}$ is normally reducible to Q in E-HA^ω.
- P is Kleene-primitive-recursive Weihrauch reducible to Q in $\widehat{\text{E-PA}}^\omega\!\upharpoonright \iff \mathrm{P}$ is normally reducible to Q in $\widehat{\text{E-HA}}^\omega\!\upharpoonright$.

Idea of the Proof.
($\Leftarrow$) is shown by using the modified realizability interpretation;
($\Rightarrow$) is shown by using the negative translation.

Meta-theorems with respect to RCA_0 and ACA_0

$\mathrm{AC}^{0,0}$: $\forall x^{\mathbb{N}} \exists y^{\mathbb{N}} A(x, y) \to \exists Y^{\mathbb{N}^{\mathbb{N}}} \forall x^{\mathbb{N}} A(x, Yx)$.

$\mathrm{QF\text{-}AC}^{0,0}$: $\mathrm{AC}^{0,0}$ where A is restricted to A_{qf}.

$\Pi^0_1\text{-}\mathrm{AC}^{0,0}$: $\mathrm{AC}^{0,0}$ where A is restricted to $\forall z^{\mathbb{N}} A_{\mathrm{qf}}$

DNS^0 : $\forall x^{\mathbb{N}} \neg\neg A \to \neg\neg \forall x^{\mathbb{N}} A$.

$\Sigma^0_1\text{-}\mathrm{DNS}^0$: DNS^0 where A is restricted to $\exists y^{\mathbb{N}} A_{\mathrm{qf}}$.

$\Sigma^0_2\text{-}\mathrm{DNS}^0$: DNS^0 where A is restricted to $\exists y^{\mathbb{N}} \forall z^{\mathbb{N}} A_{\mathrm{qf}}$.

Lemma. (Kohlenbach/F. 2018, F. 2020)

- The negative translation of an instance of $\mathrm{QF\text{-}AC}^{0,0}$ (resp. $\Pi^0_1\text{-}\mathrm{AC}^{0,0}$) is derived from $\mathrm{QF\text{-}AC}^{0,0} + \Sigma^0_1\text{-}\mathrm{DNS}^0$ (resp. $\Pi^0_1\text{-}\mathrm{AC}^{0,0} + \Sigma^0_2\text{-}\mathrm{DNS}^0$).
- The modified realizability interpretation of an instance of $\Sigma^0_1\text{-}\mathrm{DNS}^0$ (resp. $\Sigma^0_2\text{-}\mathrm{DNS}^0$) is derived from $\Sigma^0_1\text{-}\mathrm{DNS}^0 + \mathrm{QF\text{-}AC}^{0,0}$ (resp. $\Sigma^0_2\text{-}\mathrm{DNS}^0 + \Pi^0_1\text{-}\mathrm{AC}^{0,0}$).

Theorem. (F. 2020)

Let $\mathrm{P} : \forall f\, (A_1(f) \to \exists g\, B_1(f,g))$, $\mathrm{Q} : \forall f\, (A_2(f) \to \exists g B_2(f,g))$ with $\exists$-free formulas A_1, A_2, B_1, and B_2.

- P is Gödel-primitive-recursive Weihrauch reducible to Q in E-PA$^\omega$ (resp. E-PA$^\omega$ $+ \mathrm{QF}\text{-}\mathrm{AC}^{0,0}$, E-PA$^\omega$ $+ \Pi^0_1\text{-}\mathrm{AC}^{0,0}$) if and only if P is normally reducible to Q in E-HA$^\omega$ (resp. E-HA$^\omega$ $+ \mathrm{QF}\text{-}\mathrm{AC}^{0,0} + \Sigma^0_1\text{-}\mathrm{DNS}^0$, E-HA$^\omega$ $+ \Pi^0_1\text{-}\mathrm{AC}^{0,0} + \Sigma^0_2\text{-}\mathrm{DNS}^0$).
- P is Kleene-primitive-recursive Weihrauch reducible to Q in $\widehat{\text{E-PA}}^\omega\!\restriction$ (resp. $\widehat{\text{E-PA}}^\omega\!\restriction + \mathrm{QF}\text{-}\mathrm{AC}^{0,0}$, $\widehat{\text{E-PA}}^\omega\!\restriction + \Pi^0_1\text{-}\mathrm{AC}^{0,0}$) if and only if P is normally reducible to Q in $\widehat{\text{E-HA}}^\omega\!\restriction$ (resp. $\widehat{\text{E-HA}}^\omega\!\restriction + \mathrm{QF}\text{-}\mathrm{AC}^{0,0} + \Sigma^0_1\text{-}\mathrm{DNS}^0$, $\widehat{\text{E-HA}}^\omega\!\restriction + \Pi^0_1\text{-}\mathrm{AC}^{0,0} + \Sigma^0_2\text{-}\mathrm{DNS}^0$).

Observation: IVT, WKL^c, and WKL_2

- A lot of existing proofs in constructive reverse mathematics show not only provability but rather normal reducibility. However, this is not always the case.
- Here, as an example, we deal with some derivability results in constructive reverse mathematics on the intermediate value theorem IVT, the convex weak König's lemma WKL^c, and the weak König's lemma for trees having exactly 2 branches for each non-0-height WKL_2.

Observation: IVT, WKL^c, and WKL_2

- A lot of existing proofs in constructive reverse mathematics show not only provability but rather normal reducibility. However, this is not always the case.
- Here, as an example, we deal with some derivability results in constructive reverse mathematics on the intermediate value theorem IVT, the convex weak König's lemma WKL^c, and the weak König's lemma for trees having exactly 2 branches for each non-0-height WKL_2.

Theorem.(Berger/Ishihara/Kihara/Nemoto 2019)

1. $\mathsf{EL}_0 \vdash \mathrm{IVT} \to \mathrm{WKL}_2$.
2. $\mathsf{EL}_0 \vdash \mathrm{IVT} \leftrightarrow \mathrm{WKL}^c$.

Remark. All of $\mathrm{IVT}, \mathrm{WKL}^c$ and WKL_2 are formalized as Π_2 sentences in the applicable form of our meta-theorems.

Remark. All of $\mathrm{IVT}, \mathrm{WKL}^c$ and WKL_2 are formalized as Π_2 sentences in the applicable form of our meta-theorems.

Proposition.

WKL_2 is normally reducible to IVT in $\widehat{\mathsf{E\text{-}HA}}^{\omega}\!\restriction + \mathrm{QF\text{-}AC}^{0,0}$.

Proof. By inspecting the proof of $\mathsf{EL}_0 \vdash \mathrm{IVT} \to \mathrm{WKL}_2$.

Remark. All of $\mathrm{IVT}, \mathrm{WKL}^c$ and WKL_2 are formalized as Π_2 sentences in the applicable form of our meta-theorems.

Proposition.

WKL_2 is normally reducible to IVT in $\widehat{\text{E-HA}}^\omega\!\upharpoonright + \mathrm{QF\text{-}AC}^{0,0}$.

Proof. By inspecting the proof of $\mathsf{EL}_0 \vdash \mathrm{IVT} \to \mathrm{WKL}_2$.

Proposition.

IVT is normally reducible to WKL^c in $\widehat{\text{E-HA}}^\omega\!\upharpoonright + \mathrm{QF\text{-}AC}^{0,0}$.

Proof. By inspecting the proof of $\mathsf{EL}_0 \vdash \mathrm{WKL}^c \to \mathrm{IVT}$.

Remark. All of $\mathrm{IVT}, \mathrm{WKL}^c$ and WKL_2 are formalized as Π_2 sentences in the applicable form of our meta-theorems.

Proposition.

WKL_2 is normally reducible to IVT in $\widehat{\text{E-HA}}^\omega \restriction + \mathrm{QF\text{-}AC}^{0,0}$.

Proof. By inspecting the proof of $\mathrm{EL}_0 \vdash \mathrm{IVT} \to \mathrm{WKL}_2$.

Proposition.

IVT is normally reducible to WKL^c in $\widehat{\text{E-HA}}^\omega \restriction + \mathrm{QF\text{-}AC}^{0,0}$.

Proof. By inspecting the proof of $\mathrm{EL}_0 \vdash \mathrm{WKL}^c \to \mathrm{IVT}$.

Corollary.

WKL_2 (resp. IVT) is Kleene-primitive-recursive Weihrauch reducible to IVT (resp. WKL^c) in $\widehat{\text{E-PA}}^\omega \restriction + \mathrm{QF\text{-}AC}^{0,0}$ (even in $\widehat{\text{E-HA}}^\omega \restriction$ by the proof of our meta-theorem).

Remark.

On the other hand, in the proof of $\mathrm{IVT} \to \mathrm{WKL}^c$ in Berger/Ishihara/Kihara/Nemoto 2019, for a given infinite convex tree T, IVT is first used to construct an infinite convex subtree T' having at most 2 branches for each height, and then it is used again for taking an infinite path through T'.

Definition. (2-copies Generalizations)

Let $\mathrm{P} : \forall f\,(A_1(f) \to \exists g\, B_1(f,g))$, $\mathrm{Q} : \forall f\,(A_2(f) \to \exists g B_2(f,g))$.

- For a finite-type arithmetic S^ω containing E-HA^ω, P **is Gödel/Kleene-primitive-recursive Weihrauch reducible to the** 2**-copies of** Q **in** S^ω if there exist closed terms s, t, and u in $\mathbf{T}/\mathbf{T_0}$ such that S^ω proves

$$
\begin{array}{l}
\forall f (A_1(f) \to A_2(sf)) \wedge \\
\forall f, g' \,(B_2(sf, g') \wedge A_1(f) \to A_2(tfg')) \wedge \\
\forall f, g', g'' \,(B_2(tfg', g'') \wedge B_2(sf, g') \wedge A_1(f) \to B_1(f, ufg'g''))
\end{array}.
$$

- For a finite-type arithmetic S^ω containing $\widehat{\text{E-HA}}^\omega\!\upharpoonright$, P **is normally reducible to the** 2**-copies of** Q **in** S^ω if S^ω proves

$$
\forall f \left(\begin{array}{l} A_1(f) \to \\ \exists f' \left(A_2(f') \wedge \forall g' \left(\begin{array}{l} B_2(f', g') \to \\ \exists f'' \left(A_2(f'') \wedge \forall g'' \left(\begin{array}{l} B_2(f'', g'') \to \\ \exists g B_1(f, g) \end{array} \right) \right) \end{array} \right) \right) \end{array} \right).
$$

Corollary. (k-copies Generalizations)

Let $\mathrm{P} : \forall f\,(A_1(f) \to \exists g\, B_1(f,g))$, $\mathrm{Q} : \forall f\,(A_2(f) \to \exists g B_2(f,g))$ with $\exists$-free formulas A_1, A_2, B_1, and B_2.

- P is Gödel-primitive-recursive Weihrauch reducible to the k-copies of Q in E-PA$^\omega$ (resp. E-PA$^\omega$ + $\mathrm{QF\text{-}AC}^{0,0}$, E-PA$^\omega$ + $\Pi^0_1\text{-}\mathrm{AC}^{0,0}$) if and only if P is normally reducible to the k-copies of Q in E-HA$^\omega$ (resp. E-HA$^\omega$ + $\mathrm{QF\text{-}AC}^{0,0} + \Sigma^0_1\text{-}\mathrm{DNS}^0$, E-HA$^\omega$ + $\Pi^0_1\text{-}\mathrm{AC}^{0,0}$ + $\Sigma^0_2\text{-}\mathrm{DNS}^0$).
- P is Kleene-primitive-recursive Weihrauch reducible to the k-copies of Q in $\widehat{\text{E-PA}}^\omega\!\upharpoonright$ (resp. $\widehat{\text{E-PA}}^\omega\!\upharpoonright + \mathrm{QF\text{-}AC}^{0,0}$, $\widehat{\text{E-PA}}^\omega\!\upharpoonright + \Pi^0_1\text{-}\mathrm{AC}^{0,0}$) if and only if P is normally reducible to the k-copies of Q in $\widehat{\text{E-HA}}^\omega\!\upharpoonright$ (resp. $\widehat{\text{E-HA}}^\omega\!\upharpoonright + \mathrm{QF\text{-}AC}^{0,0} + \Sigma^0_1\text{-}\mathrm{DNS}^0$, $\widehat{\text{E-HA}}^\omega\!\upharpoonright + \Pi^0_1\text{-}\mathrm{AC}^{0,0}$ + $\Sigma^0_2\text{-}\mathrm{DNS}^0$).

Applying the meta-theorem for $k = 2$ to the proof of $\mathrm{IVT} \to \mathrm{WKL}^c$ in Berger/Ishihara/Kihara/Nemoto 2019, one can obtain a non-trivial result in the style of computable analysis:

Proposition.

WKL^c is Kleene-primitive-recursive Weihrauch reducible to the 2-copies of IVT in $\widehat{\text{E-PA}}^{\omega}\restriction + \mathrm{QF\text{-}AC}^{0,0}$.

Remark

- In general, for an intuitionistic finite-type arithmetic iS^ω which satisfies the deduction theorem and proves the existence of an instance of Q, iS^ω proves $\mathrm{Q} \to \mathrm{P}$ if and only if P is normally reducible to Q in $\mathrm{iS}^\omega + \mathrm{Q}$.
- In the context of computable analysis (or higher order reverse mathematics), this corresponds to the notion that there exists a Gödel/Kleene-primitive-recursive functional Φ (verifiably in S^ω) which transforms a functional providing a solution to an instance of Q to a functional providing a solution to an instance of P.

Recap

P is normally reducible to Q in iS^ω.

$$\downarrow$$

P is normally reducible to the 2-copies of Q in iS^ω.

$$\downarrow$$

P is normally reducible to the 3-copies of Q in iS^ω.

$$\downarrow$$

$$\vdots$$

$$\downarrow$$

P is normally reducible to Q in $\mathsf{iS}^\omega + \mathrm{Q}$

$$\Updownarrow$$

$$\mathsf{iS}^\omega \vdash \mathrm{Q} \to \mathrm{P}$$

Definition.(Normal Derivability)

Let $\mathrm{P} : \forall f\, (A_1(f) \to \exists g\, B_1(f,g))$, $\mathrm{Q} : \forall f\, (A_2(f) \to \exists g B_2(f,g))$. P **is normally T-derivable from** Q **in** iS^ω if there exists a closed term s in **T** such that iS^ω proves the following two:

1. $\forall f\, \big(A_1(f) \to \forall m^{\mathbb{N}} A_2(smf)\big)$;
2. $$\forall f \left(\begin{array}{l} A_1(f) \wedge \forall m^{\mathbb{N}} \left(A_2(smf) \to \exists g' B_2\left(smf, g'\right)\right) \\ \to \exists g B_1(f,g) \end{array} \right).$$

The normal $\mathbf{T_0}$-derivability is defined with using $\mathbf{T_0}$ in stead of $\mathbf{T}$.

The fact that P is normally derivable from Q (in iS^ω) demands some proof of that Q implies P with the following structure:

1. Fix f such that $A_1(f)$;
2. Assuming $A_1(f)$, derive $\exists g B_1(f,g)$ by using Q for the countably many instances which are provided primitive recursively (in the sense of Gödel/Kleene) in f.

Definition.(Normal Derivability)

Let $\mathrm{P} : \forall f\, (A_1(f) \to \exists g\, B_1(f,g))$, $\mathrm{Q} : \forall f\, (A_2(f) \to \exists g B_2(f,g))$. P **is normally T-derivable from** Q **in** iS^ω if there exists a closed term s in $\mathbf{T}$ such that iS^ω proves the following two:

1. $\forall f\, \left(A_1(f) \to \forall m^{\mathbb{N}} A_2(smf)\right);$
2. $$\forall f \left(\begin{array}{l} A_1(f) \wedge \forall m^{\mathbb{N}} \left(A_2(smf) \to \exists g' B_2\left(smf, g'\right)\right) \\ \to \exists g B_1(f,g) \end{array} \right).$$

The normal $\mathbf{T_0}$-derivability is defined with using $\mathbf{T_0}$ in stead of $\mathbf{T}$.

Remark. The normal derivability is (properly) weaker than the normal reducibility: Omitting $\forall m^{\mathbb{N}}$ from the definition of the normal derivability makes it (intuitionistically) equivalent to the normal reducibility.

Proposition. (cf. Berger/Ishihara/Schuster 2012)

WKL is normally $\mathbf{T_0}$-derivable from LLPO in $\widehat{\text{E-HA}}^{\omega}\restriction + \Pi^0_1\text{-}\mathrm{AC}^{0,0}_{\vee} + \mathrm{QF}\text{-}\mathrm{AC}^{0,0} + \Sigma^0_1\text{-}\mathrm{DNS}^0$, where

$$\Pi^0_1\text{-}\mathrm{AC}^{0,0}_{\vee} : \begin{array}{l} \forall n^{\mathbb{N}} \left(\forall x^{\mathbb{N}} A_{\mathrm{qf}}(n,x) \vee \forall y^{\mathbb{N}} B_{\mathrm{qf}}(n,y)\right) \\ \to \exists h^{\mathbb{N}\to\mathbb{N}} \forall n^{\mathbb{N}} \left(\begin{array}{l} \left(h(n) = 0 \to \forall x^{\mathbb{N}} A_{\mathrm{qf}}(n,x)\right) \wedge \\ \left(h(n) \neq 0 \to \forall y^{\mathbb{N}} B_{\mathrm{qf}}(n,y)\right) \end{array} \right). \end{array}$$

Theorem. (Parallelization)

Let $\mathrm{P} : \forall f\,(A_1(f) \to \exists g\, B_1(f,g))$, $\mathrm{Q} : \forall f\,(A_2(f) \to \exists g B_2(f,g))$ with $\exists$-free formulas A_1, A_2, B_1, and B_2.

1. P is Gödel-primitive-recursive Weihrauch reducible to $\widehat{\mathrm{Q}}$ in E-PA$^\omega$
   $\Longleftrightarrow$
   P is normally **T**-derivable from Q in E-HA$^\omega + \mathrm{AC}^{0,\omega}$
2. P is Kleene-primitive-recursive Weihrauch reducible to $\widehat{\mathrm{Q}}$ in $\widehat{\text{E-PA}}^\omega\!\restriction$
   $\Longleftrightarrow$
   P is normally $\mathbf{T_0}$-derivable from Q in $\widehat{\text{E-HA}}^\omega\!\restriction + \mathrm{AC}^{0,\omega}$

$\mathrm{AC}^{0,\omega}$(countable choice) : $\forall x^{\mathbb{N}} \exists f^\tau A(x,f) \to \exists F^{\mathbb{N}\to\tau} \forall x^{\mathbb{N}} A(x, Fx)$.

Theorem. (Parallelization)

Let $\mathrm{P} : \forall f\,(A_1(f) \to \exists g\, B_1(f,g))$, $\mathrm{Q} : \forall f\,(A_2(f) \to \exists g B_2(f,g))$ with $\exists$-free formulas A_1, A_2, B_1, and B_2.

1. P is Gödel-primitive-recursive Weihrauch reducible to $\widehat{\mathrm{Q}}$ in E-PA$^\omega$ (resp. E-PA$^\omega + \mathrm{QF\text{-}AC}^{0,0}$, E-PA$^\omega + \Pi^0_1\text{-}\mathrm{AC}^{0,0}$)
$\iff$
P is normally **T**-derivable from Q in E-HA$^\omega + \mathrm{AC}^{0,\omega}$ (resp. E-HA$^\omega + \mathrm{AC}^{0,\omega} + \Sigma^0_1\text{-}\mathrm{DNS}^0$, E-HA$^\omega + \mathrm{AC}^{0,\omega} + \Sigma^0_2\text{-}\mathrm{DNS}^0$).

2. P is Kleene-primitive-recursive Weihrauch reducible to $\widehat{\mathrm{Q}}$ in $\widehat{\text{E-PA}}^\omega\!\restriction$ (resp. $\widehat{\text{E-PA}}^\omega\!\restriction + \mathrm{QF\text{-}AC}^{0,0}$, $\widehat{\text{E-PA}}^\omega\!\restriction + \Pi^0_1\text{-}\mathrm{AC}^{0,0}$)
$\iff$
P is normally $\mathbf{T_0}$-derivable from Q in $\widehat{\text{E-HA}}^\omega\!\restriction + \mathrm{AC}^{0,\omega}$ (resp. $\widehat{\text{E-HA}}^\omega\!\restriction + \mathrm{AC}^{0,\omega} + \Sigma^0_1\text{-}\mathrm{DNS}^0$, $\widehat{\text{E-HA}}^\omega\!\restriction + \mathrm{AC}^{0,\omega} + \Sigma^0_2\text{-}\mathrm{DNS}^0$).

$\mathrm{AC}^{0,\omega}$(countable choice) : $\forall x^{\mathbb{N}} \exists f^\tau A(x,f) \to \exists F^{\mathbb{N}\to\tau} \forall x^{\mathbb{N}} A(x, Fx)$.

A Example: WKL and LLPO

There seem to be many results (proofs) in computable analysis and constructive reverse mathematics to which our meta-theorems are applicable.

By applying (the proof of) our meta-theorem, the following two results are shown to be equivalent:

Proposition. (cf. Berger/Ishihara/Schuster 2012)

WKL is normally $\mathbf{T_0}$-derivable from LLPO in $\widehat{\textsf{E-HA}}^{\omega}\!\restriction + \Pi^0_1\text{-}\mathrm{AC}^{0,0}_{\vee} + \mathrm{QF}\text{-}\mathrm{AC}^{0,0} + \Sigma^0_1\text{-}\mathrm{DNS}^0$.

Proposition. (cf. Brattka/Gherardi 2011)

WKL is Kleene-primitive-recursive Weihrauch reducible to $\widehat{\mathrm{LLPO}}$ in $\widehat{\textsf{E-PA}}^{\omega}\!\restriction + \mathrm{QF}\text{-}\mathrm{AC}^{0,0}$.

Observation

- The direct proofs of the previous two propositions are somewhat similar (using Π^0_1-IND for the verifications).
- Nevertheless, the primitive recursive witnesses for the latter is not obvious from the proof of the former.
- On the other hand, the proof of the latter heavily uses classical logic for the verification, and hence, the former is also not an immediate consequence from the latter.
- Thus our meta-theorems should give rise to new results in one of the contexts from the results (proofs) in the other context.

Remark.

In our meta-theorems for parallelizations, the countable choice $\mathrm{AC}^{0,\omega}$ is crucial (it cannot be replaced by $\mathrm{QF\text{-}AC}^{0,0}$):

If $\mathrm{AC}^{0,\omega}$ be replaced by $\mathrm{QF\text{-}AC}^{0,0}$, since WKL is Kleene-primitive-recursive Weihrauch reducible to $\widehat{\mathrm{LLPO}}$ in $\widehat{\text{E-PA}}^{\omega}\!\restriction + \mathrm{QF\text{-}AC}^{0,0}$, we have that WKL is normally $\mathbf{T_0}$-derivable from LLPO in $\widehat{\text{E-HA}}^{\omega}\!\restriction + \mathrm{QF\text{-}AC}^{0,0} + \Sigma^0_1\text{-}\mathrm{DNS}^0$, and hence, WKL is provable in $\widehat{\text{E-PA}}^{\omega}\!\restriction + \mathrm{QF\text{-}AC}^{0,0}$. This is a contradiction.

Recap

P is normally reducible to Q in iS^ω.	$\Longrightarrow$	P is normally derivable from Q in $\mathsf{iS}^\omega + \mathrm{AC}^{0,\omega}$
$\downarrow$		
P is normally reducible to the 2-copies of Q in iS^ω.		
$\downarrow$		$\downarrow$
P is normally reducible to the 3-copies of Q in iS^ω.		
$\downarrow$		$\downarrow$
$\vdots$		$\vdots$
P is normally reducible to Q in $\mathsf{iS}^\omega + \mathrm{Q}$		
$\Updownarrow$		
$\mathsf{iS}^\omega \vdash \mathrm{Q} \to \mathrm{P}$	$\Longrightarrow$	$\mathsf{iS}^\omega + \mathrm{AC}^{0,\omega} \vdash \mathrm{Q} \to \mathrm{P}$

Syntactical Restriction

Remark.

- Our meta-theorems are applicable only for existence statements formalized in finite-type arithmetic with $\exists$-free formulas. Of course, there are many mathematical statements to which our meta-theorems are not applicable (e.g. the Bolzano–Weierstrass theorem).
- There is a counterexample which shows that our meta-theorems does not hold already for

$$\forall f^{\mathbb{N}^{\mathbb{N}}} \left(\exists x^{\mathbb{N}} \forall y^{\mathbb{N}} A_{\mathrm{qf}}(f, x, y) \to \exists g^{\mathbb{N}^{\mathbb{N}}} \forall z^{\mathbb{N}} B_{\mathrm{qf}}(f, g, z) \right).$$

References

1. J. Berger, H. Ishihara, T. Kihara and T. Nemoto, The binary expansion and the intermediate value theorem in constructive reverse mathematics, Arch. Math. Logic 58, pp. 203–217, 2019.
2. J. Berger, H. Ishihara and P. Schuster, The weak König lemma, Brouwer's fan theorem, de Morgan's law, and dependent choice, Rep. Math. Logic 47, pp. 63–86, 2012.
3. V. Brattka and Gherardi, Weihrauch degrees, omniscience principles and weak computability, J. Symbolic Logic 76(1), pp. 143–176, 2011.
4. M. Fujiwara, Weihrauch and constructive reducibility between existence statements, Computability, Pre-press, 2020.
5. M. Fujiwara, Parallelizations in Weihrauch reducibility and constructive reverse mathematics, LNCS 12098, pp. 38-49, 2020.

Thank you!