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On the descriptive complexity of Salem sets

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Joint work with Alberto Marcone

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Question

During the IMS Graduate Summer School in Logic in 2018, Slaman asked:

"What is the descriptive complexity of the family of closed Salem sets in $[0, 1]$?"

Hausdorff dimension

Standard notion in geometric measure theory.

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Using Frostman's lemma, the Hausdorff dimension of $A \in \mathcal{B}(\mathbb{R}^d)$ can be written as

$$\dim_{\mathcal{H}}(A) = \sup\{s : (\exists \mu \in \mathbb{P}(A)) (\exists c > 0) (\forall x \in \mathbb{R}^d) (\forall r > 0) \\ (\mu(B(x,r)) \leq cr^s)\}$$

In other words, it coincides with the capacitary dimension.

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Fourier transform of μ: $\widehat{\mu}\colon \mathbb{R}^d \to \mathbb{C}$ defined as

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The **Fourier dimension** of $A \subset \mathbb{R}^d$ is defined as

$$\begin{aligned}\dim_{\mathrm{F}}(A) := \sup\{s \in [0, d] \,:\, &(\exists \mu \in \mathbb{P}(A))\,(\exists c > 0)\,(\forall x \in \mathbb{R}^d)\\ &(|\widehat{\mu}(x)| \le c|x|^{-s/2})\}\end{aligned}$$

Salem sets

Proposition (Folklore?)

For every $A \in \mathcal{B}(\mathbb{R}^d)$ *we have* $\dim_{\mathrm{F}}(A) \leq \dim_{\mathcal{H}}(A)$

This shows that the Fourier dimension can be used to obtain lower bounds for the Hausdorff dimension.

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This shows that the Fourier dimension can be used to obtain lower bounds for the Hausdorff dimension.

A set A s.t. $\dim_{\mathcal{H}}(A) = \dim_{\mathrm{F}}(A)$ is called **Salem set**.

Salem sets

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Deterministic (non-trivial) Salem sets are rare.
Classic example: Jarník's fractal.

Theorem (Jarník 1928, Besicovitch 1934, Kaufmann 1981)

For every $\alpha \geq 0$

$$\dim(\{x \in [0,1] \,:\, x \textit{ is } \alpha\textit{-well approximable}\}) = \frac{2}{2+\alpha}$$

Wadge reducibility

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Let X and Y be Polish spaces.

$A \subset X$ is **Wadge reducible** to $B \subset Y$ ($A \leq_W B$) if there is a continuous map $f\colon X \to Y$ s.t.

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B is called $\mathbf{\Gamma}$-hard if, for every $A \in \mathbf{\Gamma}(2^{\mathbb{N}})$, $A \leq_W B$.
B is called $\mathbf{\Gamma}$-complete if it is $\mathbf{\Gamma}$-hard and $B \in \mathbf{\Gamma}(Y)$

Salem sets in $[0, 1]$

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Lemma (Marcone, Reimann, Slaman, V.)

- $\{(A,p) \in \mathbf{K}([0,1]) \times [0,1] \,:\, \dim_{\mathcal{H}}(A) > p\}$ *is* $\mathbf{\Sigma}^0_2$*;*
- $\{(A,p) \in \mathbf{K}([0,1]) \times [0,1] \,:\, \dim_{\mathcal{H}}(A) \geq p\}$ *is* $\mathbf{\Pi}^0_3$*;*

- $\{(A,p) \in \mathbf{K}([0,1]) \times [0,1] \,:\, \dim_{\mathrm{F}}(A) > p\}$ *is* $\mathbf{\Sigma}^0_2$*;*
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The proof relies on the compactness of the ambient space $[0,1]$.

Salem sets in $[0,1]$

Lemma (Marcone, Reimann, Slaman, V.)

For every $p \in [0,1]$ there is a continuous (in fact computable) map $f_p \colon 2^{\mathbb{N}} \to \mathscr{S}([0,1])$ s.t.

$$\dim(f_p(x)) = \begin{cases} p & \text{if } x \in Q_2 \\ 0 & \text{if } x \notin Q_2 \end{cases}$$

where $Q_2 = \{x \in 2^{\mathbb{N}} : (\forall^\infty n)(x(n) = 0)\}$ is $\boldsymbol{\Sigma}^0_2$-complete.

Salem sets in $[0,1]$

Theorem (Marcone, Reimann, Slaman, V.)

The sets

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Problem: the Fourier dimension is sensitive to the ambient space.

If a set A is contained in a m-dimensional hyperplane (with $m < d$) then $\dim_{\mathrm{F}}(A) = 0$.

Some "curvature" is necessary to have positive Fourier dimension.

Solutions?

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We can exploit a "higher-dimensional analogue" of Jarník's fractal, recently defined by Fraser and Hambrook [4].

Theorem (Fraser, Hambrook)

For every $\alpha \geq 0$, the set $E(K, B, \alpha)$ is a Salem set of dimension $2d/(2+\alpha)$.

Salem sets in $[0,1]^d$

Similarly to the one-dimensional case we have:

Lemma (Marcone, V.)

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Theorem (Marcone, V.)

For every $d \geq 1$, the sets

$$\{(A,p) \in \mathbf{K}([0,1]^d) \times [0,d) \,:\, \dim_{\mathcal{H}}(A) > p\},$$
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Both Hausdorff and Fourier dimensions are preserved when moving from $[0,1]^d$ to $\mathbb{R}^d$.

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Hardness results (lower bounds) are corollaries, while upper bounds are more delicate.

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- Not metrizable if the ambient space is not compact	$\pm$ Coarser than Vietoris topology

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There is $G = \bigcup_n K_n$ with $\dim_{\mathrm{F}}(G) = 1$ and $\dim_{\mathrm{F}}(K_n) = 0$.

Theorem (Marcone, V.)

For every pointclass $\boldsymbol{\Gamma}$ *and every non-empty* $A \in \boldsymbol{\Gamma}(\mathbb{R}^d)$,

$$\dim_{\mathrm{F}}(A) = \sup\{\dim_{\mathrm{F}}(K) \,:\, K \subsetneq A \text{ is bounded and } K \in \boldsymbol{\Gamma}(\mathbb{R}^d)\}.$$

Salem sets in $\mathbb{R}^d$

Theorem (Marcone, V.)

Fix $d \geq 1$. For every $p < d$ the sets

$$\{A \in \mathbf{F}(\mathbb{R}^d) : \dim_{\mathcal{H}}(A) > p\},$$
$$\{A \in \mathbf{F}(\mathbb{R}^d) : \dim_{\mathrm{F}}(A) > p\}$$

are $\mathbf{\Sigma}^0_2$-complete. Moreover, for every $q > 0$ the sets

$$\{A \in \mathbf{F}(\mathbb{R}^d) : \dim_{\mathcal{H}}(A) \geq q\},$$
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Effectivizations

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All the results obtained are actually effective:

Theorem (Marcone, V.)

Let X be $[0,1]^d$ or $\mathbb{R}^d$ for some $d \geq 1$.

- $\{(A,p) \in \mathbf{F}(X) \times [0,d) : \dim_{\mathcal{H}}(A) > p\}$ *is* Σ^0_2*-complete*
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Effective measurability

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Our results imply that the maps $\dim_{\mathcal{H}}$ and $\dim_{\mathrm{F}}$ are effectively Σ^0_3-measurable.

Weihrauch degree of $\dim_{\mathrm{F}}$

Theorem (Brattka [1])

Let lim *be the problem of finding the limit in the Baire space.*

$$f \text{ is effectively } \Sigma^0_{k+1}\text{-measurable} \iff f \le_{\mathrm{W}} \mathsf{lim}^{[k]}$$

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Theorem (Marcone, V.)

$\dim_{\mathrm{F}} \equiv_{\mathrm{W}} \mathsf{lim}^{[2]}$

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