Computability and adjunctions of Euclidean spaces

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Computable topological spaces

Computable topological space is a triple $(X, \mathcal{T}, (I_i))$ where

- $(X, \mathcal{T})$ is a topological space
- $\{I_i \mid i \in \mathbb{N}\}$ is a base for $\mathcal{T}$
- there exist computably enumerable sets $\mathcal{C}$ and $\mathcal{D}$ such that:
  1. $(i, j) \in \mathcal{C} \;\Rightarrow\; I_i \subseteq I_j$;
  2. $(i, j) \in \mathcal{D} \;\Rightarrow\; I_i \cap I_j = \emptyset$;
  3. $x \in I_i \cap I_j \;\Rightarrow\; \exists k(x \in I_k$ and $(k, i), (k, j) \in \mathcal{C})$;
  4. $x \neq y \;\Rightarrow\; \exists i, j(x \in I_i,\; y \in I_j$ and $(i, j) \in \mathcal{D})$.

Computable topological spaces

We denote the finite unions of basis elements I_i by

$$J_j = \bigcup_{i \in [j]} I_i.$$

Here $j \mapsto [j]$ is a function $\mathbb{N} \to \mathcal{P}(\mathbb{N})$ such that the set $\{(i, j) \mid i \in [j]\} \subseteq \mathbb{N}^2$ is computable and there exists a computable function $\phi : \mathbb{N} \to \mathbb{N}$ such that $i \leq \phi(j)$ for each $i \in [j]$.

There exist c.e. sets $C, D \subseteq \mathbb{N}^2$ such that

- $(i, j) \in C$ implies $J_i \subseteq J_j$;
- $(i, j) \in D$ implies $J_i \cap J_j \neq \emptyset$.

Computable and semicomputable sets

Let S be a set in a computable topological space $(X, \mathcal{T}, (I_i))$.

- S is **computably enumerable** if it is closed and

$$\{i \in \mathbb{N} \mid I_i \cap S \neq \emptyset\}$$

  is c.e.

- S is **semicomputable** if it is compact and

$$\{j \in \mathbb{N} \mid S \subseteq J_j\}$$

  is c.e.

- S is **computable** if S is semicomputable and computably enumerable.

Local computable enumerability

- A is **computably enumerable up to** B if there exists c.e. set $\Omega \subseteq \mathbb{N}$ such that

$$I_i \cap A \neq \emptyset \quad \Rightarrow \quad i \in \Omega$$

and

$$i \in \Omega \quad \Rightarrow \quad I_i \cap B \neq \emptyset.$$

- S is **computably enumerable at** x if there exists a neighborhood U of x in S such that U is computably enumerable up to S.
- S is **locally computably enumerable** if S is computably enumerable at every point $x \in S$.

Local computable enumerability

- S is computably enumerable iff S is computably enumerable up to S
- If A_i is computably enumerable up to B_i for each $i = 1, \ldots, n$, then $A_1 \cup \cdots \cup A_n$ is computably enumerable up to $B_1 \cup \cdots \cup B_n$.
- A compact set is computably enumerable iff it is locally computably enumerable

Computable type

General question:

Under which (topological) conditions does the implication

$$S \text{ semicomputable} \quad \Rightarrow \quad S \text{ computable}$$

hold for a set S in a computable topological space?

$$S \text{ semicomputable and } \boxed{S \cong A} \quad \Rightarrow \quad S \text{ computable}$$

Topological space A has **computable type** if the implication above holds whenever S is homeomorphic to A.

Local computable enumerability and computable type

The following implications are equivalent:

$$S \text{ semicomputable} \quad \Rightarrow \quad S \text{ computable,} \tag{1}$$

$$S \text{ semicomputable} \quad \Rightarrow \quad S \text{ computably enumerable,} \tag{2}$$

$$S \text{ semicomputable} \quad \Rightarrow \quad S \text{ locally computably enumerable.} \tag{3}$$

(3) motivates a more *local approach* to computable type – it is sufficient to study implications of the form

$$S \text{ semicomputable and } x \in S \; \boxed{} \quad \Rightarrow \quad S \text{ c.e. at } x$$

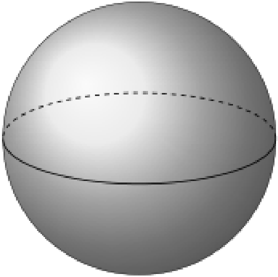
Theorem (Iljazović and Sušić, 2018, [4])

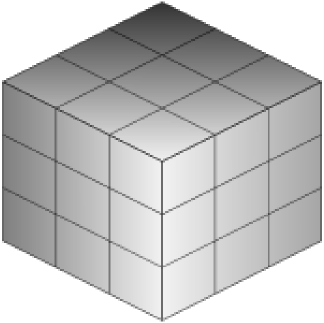
Let $(X, \mathcal{T}, (I_i))$ be a computable topological space and let S be semicomputable set in this space and let $x \in S$. Suppose there exists a neighborhood of x in S which is homeomorphic to some $\mathbb{R}^n$. Then S is c.e. at x.

Theorem (Iljazović and Sušić, 2018, [4])

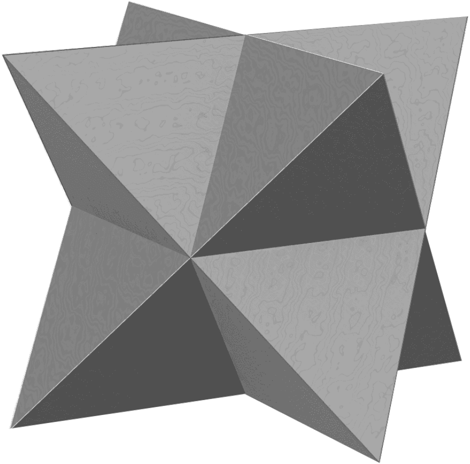
Let $(X, \mathcal{T}, (I_i))$ be a computable topological space and let S be a semicomputable set in this space which is, as a subspace of $(X, \mathcal{T})$, a manifold. Then S is a computable set in $(X, \mathcal{T}, (I_i))$.

Topological manifolds have computable type

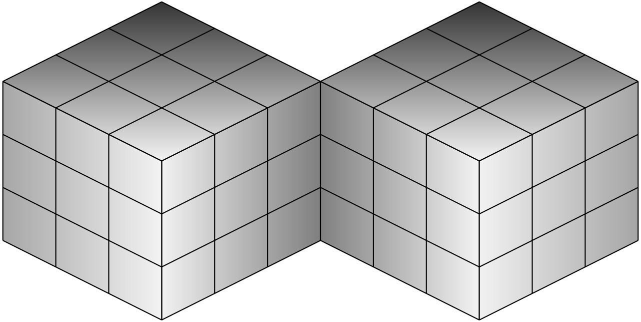




Topological manifolds have computable type



Adjunctions of manifolds?



Adjunction spaces

Let X and Y be topological spaces, let A be a subspace of X and let $f : A \to Y$ be a continuous function. Let $X \sqcup Y$ be the disjoint union of X and Y and let $\imath_X : X \to X \sqcup Y$ and $\imath_Y : Y \to X \sqcup Y$ be the cannonical inclusion maps. Let $\sim$ be an equivalence relation on $X \sqcup Y$ generated by

$$\imath_X(a) \sim \imath_Y(f(a)), \qquad \forall a \in A.$$

We denote the quotient space ${X \sqcup Y}/{\sim}$ by $X \cup_f Y$ and we call it **adjunction space** obtained by adjoining X onto Y by way of f.

(In this talk, we consider only the case where $f : A \to Y$ is an embedding.)

Proposition

Let $n \geq 2$ *and let* A *be a closed subset of* $\mathbb{R}^{n-1} \times \{0\}$ *such that* $(0, \ldots, 0) \in A$. *Let* Y *be a locally compact topological space and let* $\gamma : A \to Y$ *be an embedding such that* $\gamma(A)$ *is closed in* Y. *Suppose* $(X, \mathcal{T}, (I_i))$ *is a computable topological space and* $f : \mathbb{R}^n \cup_\gamma Y \to X$ *is an embedding such that* $f(\mathbb{R}^n \cup_\gamma Y)$ *is an open subset of a semicomputable set* S. *Then the set* $f([-1, 1]^n \cup_\gamma \emptyset)$ *is c.e. up to* S.

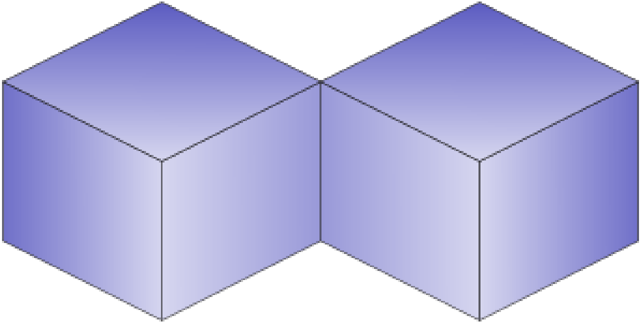
Theorem

Let $m, n \in \mathbb{N}$, $m, n \geq 2$. *Let* A *be a sufficiently thin set in an m-manifold* M *and let* B *be a sufficiently thin set in an n-manifold* N. *Let* $\gamma : A \to B$ *be a homeomorphism. Let* S *be a semicomputable set in a computable topological space* $(X, \mathcal{T}, (I_i))$ *and let* $f : M \cup_\gamma N \to S$ *be a homeomorphism. Then* S *is computable at* x *for any* $x \in f(A \cup_\gamma B)$.

Theorem

Let $m, n \in \mathbb{N}$, $m, n \geq 2$. *Let* A *be a sufficiently thin set in an m-manifold* M *and let* B *be a sufficiently thin set in an n-manifold* N. *If* $\gamma : A \to B$ *is a homeomorphism, then the adjunction space* $M \cup_\gamma N$ *has computable type.*

Examples



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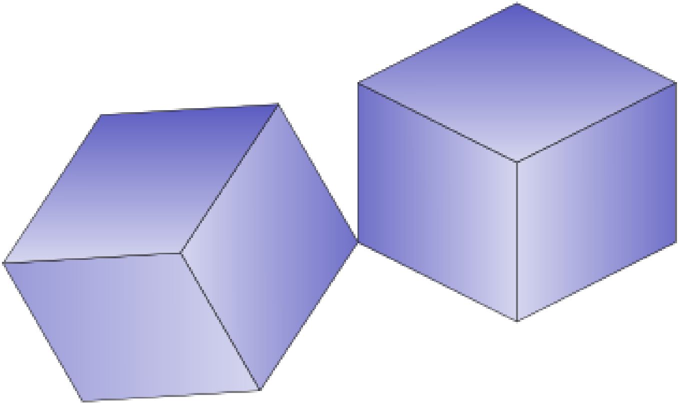

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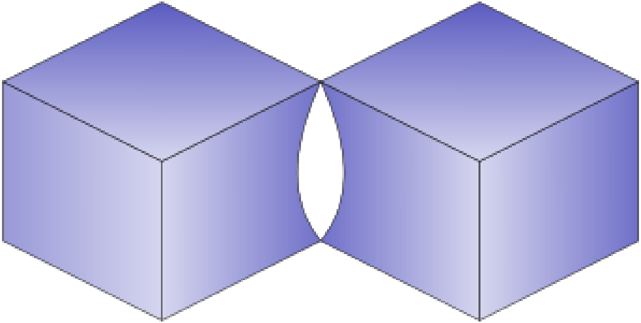

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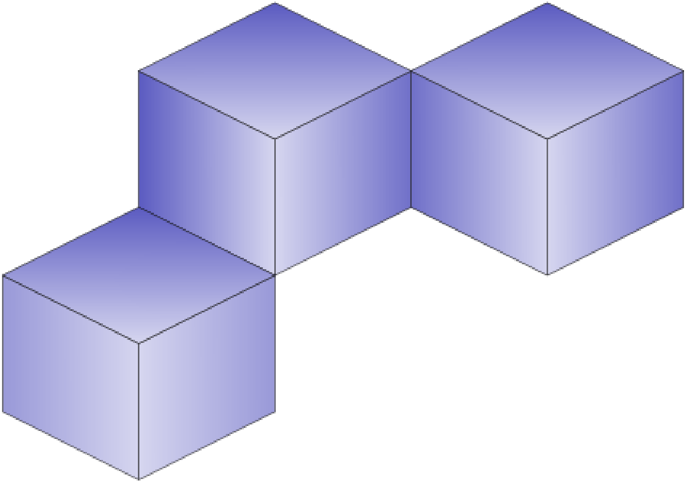

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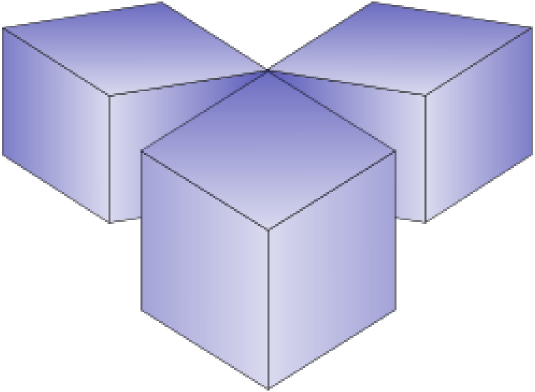
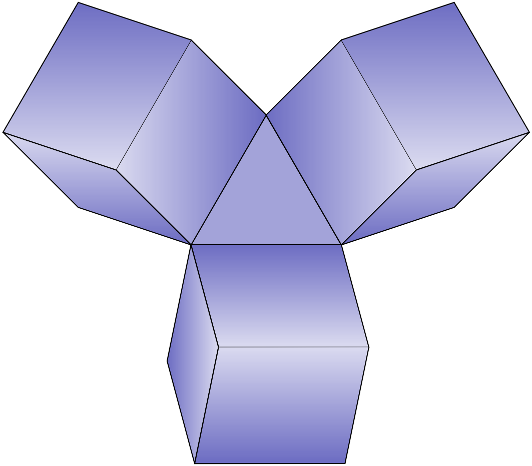


Exceptions



Further research prompts

- Manifolds with boundaries

Theorem

If S is an n-manifold with boundary ∂S in a computable topological space $(X, \mathcal{T}, (I_i))$ such that S and ∂S are semicomputable sets, then S is computable.

  - How do we modify boundary conditions for spaces obtained by attaching manifolds along subsets of their boundaries?

- More general ambient spaces
  - Can we have similar results for subsets of spaces which are not necessarily effectively T_2?

References

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