Descriptive complexity on non-Polish spaces

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DST outside Polish spaces

Descriptive Set Theory (DST):

- Mainly on Polish spaces (completely metrizable spaces).

Theoretical Computer Science induces other spaces:

- Partial functions,
- Higher-order functionals, e.g. $(\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$,
- Computation with advice,
- etc.

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Need to develop DST outside Polish spaces:

- Domains **[Selivanov]**, quasi-Polish spaces **[de Brecht]**
- Represented spaces **[Brattka, de Brecht, Pauly, Schröder, Selivanov]**

Two measures of complexity

In an admissibly represented space X, two measures of complexity of a set $A \subseteq X$.

Topological complexity

Complexity of describing A from open sets.

Symbolic complexity

Complexity of testing whether a point belongs to A.

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In an admissibly represented space X, two measures of complexity of a set $A \subseteq X$.

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Complexity of describing A from open sets.

Symbolic complexity

Complexity of testing whether a point belongs to A.

Theorem (de Brecht, 2013)

They coincide on countably-based spaces.

What about other spaces?

Motivating example

Let $A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \subseteq \mathbb{R}$.

How complicated is A?

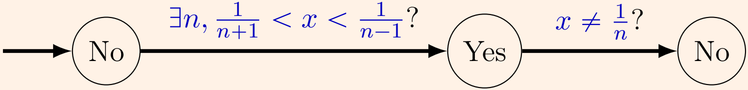
Two approaches:

- How to test $x \in A$ it with an **algorithm**?
- How to **describe** A in terms of simpler sets?

Motivating example

Let $A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \subseteq \mathbb{R}$.

Algorithm



Description

$$A = (0, +\infty) \setminus \bigcup_n \left(\frac{1}{n+1}, \frac{1}{n} \right).$$

Motivating example

These approaches are equivalent: for any $A \subseteq \mathbb{R}$,

$$A \text{ is decidable with } \leqslant 2 \text{ mind changes No-Yes-No}$$
$$\Longleftrightarrow$$
$$A \text{ is a difference of two effective open sets } (A \in \mathrm{D}_2(\mathbb{R})).$$

More generally

Are these two approaches always equivalent?

- Algorithms make sense on represented spaces,
- Descriptions using open sets make sense on topological spaces.

So let's work on topological spaces with an admissible representation.

Polynomials

Representation

A polynomial $P \in \mathbb{R}[X]$ is represented by:

- Some $n \geqslant \deg(P)$,
- The coefficients of $P = p_0 + p_1 X + \ldots + p_n X^n$.

Polynomials

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$$A = \left\{ P \in \mathbb{R}[X] : p_0 = 0 \text{ or } p_0 > \frac{1}{\deg(P)} \right\}?$$

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How complicated is

$$A = \left\{ P \in \mathbb{R}[X] : p_0 = 0 \text{ or } p_0 > \frac{1}{\deg(P)} \right\}?$$

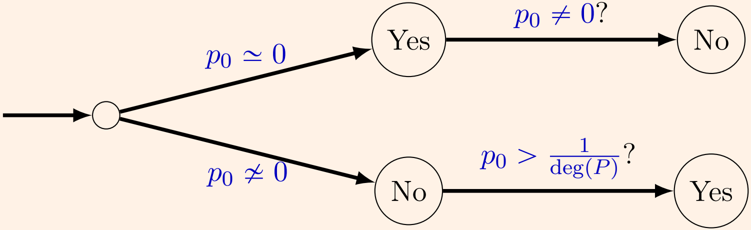
- Decidable with 2 mind-changes,
- But **not** a difference of two open sets!

Polynomials

$$A = \left\{ P \in \mathbb{R}[X] : p_0 = 0 \text{ or } p_0 > \frac{1}{\deg(P)} \right\}.$$

Algorithm

Given P and $n \geqslant \deg(P)$,



Polynomials

$$A = \left\{ P \in \mathbb{R}[X] : p_0 = 0 \text{ or } p_0 > \frac{1}{\deg(P)} \right\}.$$

Descriptive complexity

A is not a difference of 2 open sets:

$$\begin{array}{ccccc} \frac{1}{n} + \frac{X^{n+1}}{p} & \xrightarrow[p\to\infty]{} & \frac{1}{n} & \xrightarrow[n\to\infty]{} & 0 \\ \in A & & \notin A & & \in A \end{array}$$

The problem

Algorithms and topology induce the same complexity on $\mathbb{R}$ but not on $\mathbb{R}[X]$.

- Why?
- What about other spaces?
- What about other complexity levels ($\utilde{\boldsymbol{\Sigma}}^0_\alpha$, etc.)
- What do algorithms measure?

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Algorithms and topology induce the same complexity on $\mathbb{R}$ but not on $\mathbb{R}[X]$.

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- What do algorithms measure?

Guess
Algorithms reflect the **sequential** rather than **topological** aspects of the space.

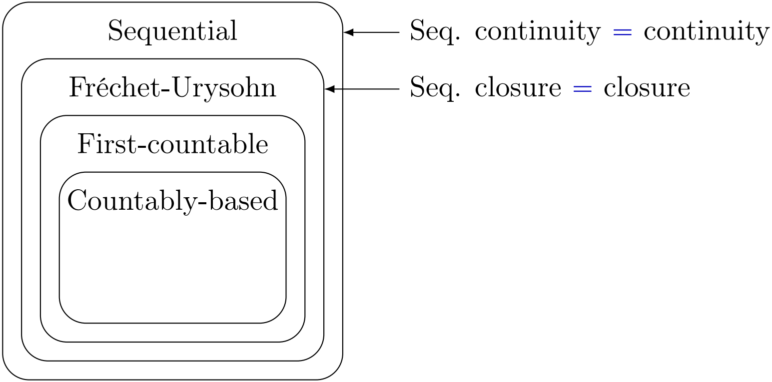
Algorithms prefer sequences

Only sequential spaces can be handled by representations (Schröder).

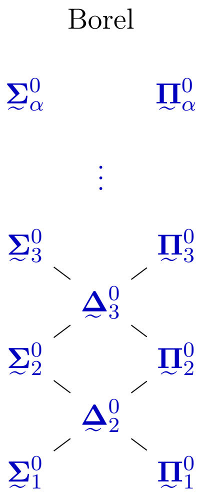
Franklin 65	Sequential spaces $\equiv$ quotients of metric spaces,
Schröder 02	Adm. rep. spaces $\equiv$ quotients of *countably-based* metric spaces.

- A **subspace** of a represented space is not a topological subspace but its sequentialization,
- A **product** of represented spaces is not the topological product but its sequentialization.

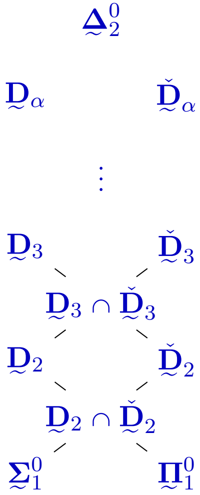
Topology vs sequences



Topological complexity



(a) Borel hierarchy



(b) Hausdorff difference hierarchy

Symbolic complexity

We work on represented spaces with a total admissible representation (X, δ_X).
Let Γ be some complexity class ($\boldsymbol{\Sigma}^0_\alpha, \Sigma^0_\alpha$, etc.).

Definition (Symbolic complexity)

A set $A \subseteq X$ belongs to $[\Gamma]$ if

$$\delta_X^{-1}(A) \in \Gamma(\mathcal{N}).$$

One always has

$$\Gamma \subseteq [\Gamma], \qquad (\delta_X \text{ is continuous})$$
$$\boldsymbol{\Sigma}^0_1 = [\boldsymbol{\Sigma}^0_1] \qquad (\text{final topology}).$$

Countably-based spaces

Symbolic and topological complexity coincide on countably-based spaces.

Theorem (De Brecht, 2013)

If X is countably-based, then $[\Gamma] = \Gamma$.

Already in (Brattka 2005), (Saint-Raymond 2007) for Polish spaces.

Theorem

The following are equivalent:

- X *is countably-based,*
- $[\underset{\sim}{\mathbf{D}}_2] = \underset{\sim}{\mathbf{D}}_2$ *in a uniform way.*

Symbolic complexity

We have seen that on $\mathbb{R}[X]$,

$$[\utilde{\mathbf{D}}_2] \nsubseteq \utilde{\mathbf{D}}_2,$$

and even

$$[\mathrm{D}_2] \nsubseteq \utilde{\mathbf{D}}_2,$$

witnessed by

$$A = \left\{ P \in \mathbb{R}[X] : p_0 = 0 \text{ or } p_0 > \frac{1}{\deg(P)} \right\}.$$

We mainly study two classes of spaces:

- CoPolish spaces $\equiv$ inductive limits of compact metric spaces,
- Spaces of open subsets of Polish spaces.

Hardness

- In a Polish space, how to show that a set A is not $\underset{\sim}{\mathbf{\Sigma}}^0_2$?

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▶ Prove that it is $\utilde{\mathbf{\Pi}}^0_2$-hard: every $\utilde{\mathbf{\Pi}}^0_2$-subset of $\mathbb{N}^\mathbb{N}$ is continuously reducible to A.

Hardness

- In a Polish space, how to show that a set A is not $\utilde{\mathbf{\Sigma}}_2^0$?

▶ Prove that it is $\utilde{\mathbf{\Pi}}_2^0$-hard: every $\utilde{\mathbf{\Pi}}_2^0$-subset of $\mathbb{N}^\mathbb{N}$ is continuously reducible to A.

Theorem (Wadge)

Let $\Gamma \neq \check{\Gamma}$. For any Borel subset A of a Polish space,

$$A \notin \Gamma \iff A \text{ is } \check{\Gamma}\text{-hard.}$$

Applies to $\Gamma = \utilde{\mathbf{D}}_\alpha, \utilde{\mathbf{\Sigma}}_\alpha^0, \utilde{\mathbf{\Pi}}_\alpha^0$, but not $\utilde{\mathbf{\Delta}}_\alpha^0$.

Hardness outside Polish spaces

- What about non-Polish spaces?

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Proposition

Let $\Gamma \neq \check{\Gamma}$. For any Borel set A,

$$A \notin [\Gamma] \iff A \text{ is } \check{\Gamma}\text{-hard.}$$

How to capture *topological* complexity?

Hardness*

Definition

$A \subseteq X$ is Γ**-hard*** if for every countably-based weaker topology τ, A is Γ-hard in (X, τ).

Hardness reflects topological complexity:

Theorem

For any Borel set $A \subseteq X$, and $\Gamma \neq \check{\Gamma}$,

$$A \notin \Gamma \iff A \text{ is } \check{\Gamma}\text{-hard*}.$$

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For any Borel set $A \subseteq X$, *and* $\Gamma \neq \check{\Gamma}$,

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$$A \notin [\Gamma] \iff A \text{ is } \check{\Gamma}\text{-hard}$$

CoPolish spaces

CoPolish space: inductive limit of (locally) compact metrizable spaces (Schröder, 2004).

$$X = \bigcup_{n \in \mathbb{N}} X_n \text{ with } X_n \subseteq X_{n+1}.$$

Example

The space X of real polynomials, with $X_n = \{\text{polynomials of degree} \leqslant n\}$.

CoPolish spaces

Topology

A set $U \subseteq X$ is open if each $U \cap X_n$ is open in X_n.

Representation

A name for $x \in X$ is given by:

- Any $n \in \mathbb{N}$ such that $x \in X_n$,
- A Cauchy name of x in X_n.

CoPolish spaces

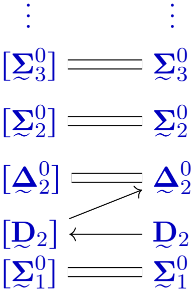

Figure: On a CoPolish space

CoPolish spaces

For all $k \in \mathbb{N}$,

$$[\underset{\sim}{\mathbf{\Sigma}}^0_k] = \underset{\sim}{\mathbf{\Sigma}}^0_k, \qquad [\underset{\sim}{\mathbf{\Delta}}^0_k] = \underset{\sim}{\mathbf{\Delta}}^0_k,$$
$$[\Sigma^0_k] = \Sigma^0_k, \qquad [\Delta^0_k] = \Delta^0_k.$$

Proof.

The degree is limit-computable from the coefficients.
So for levels $k \geqslant 2$, the space is like a countably-based space. □

But in $\mathbb{R}[X]$,

some $A \in [\mathrm{D}_2]$ is $\underset{\sim}{\mathbf{\Delta}}^0_2$-complete*.

Polynomials

Let

$$B = \left\{ \frac{1}{k_1} + \frac{X^{k_1}}{k_2} + \frac{X^{k_2}}{k_3} + \ldots + \frac{X^{k_{n-2}}}{k_{n-1}} + \frac{X^{k_{n-1}}}{k_n} : k_1 < k_2 < \ldots < k_n \text{ and } n \text{ even} \right\}.$$

One has

$$A \in [\underset{\sim}{\mathbf{D}}_2] \text{ but } A \text{ is } \underset{\sim}{\mathbf{\Delta}}^0_2\text{-complete*}.$$

CoPolish spaces

Theorem

If $\mathbf{X}$ *is coPolish, then*

$$[\undertilde{\mathbf{D}}_2] = \undertilde{\mathbf{D}}_2 \iff \mathbf{X} \text{ is Fréchet-Urysohn.}$$

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If $\mathbf{X}$ *is Hausdorff admissibly represented, then*

$$[\underset{\sim}{\mathbf{D}}_2] = \underset{\sim}{\mathbf{D}}_2 \implies \mathbf{X} \text{ is Fréchet-Urysohn.}$$

We will see that it fails for some non-Hausdorff $\mathbf{X}$.

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Proof of $\implies$.

If $\mathbf{X}$ is not Fréchet-Urysohn, then it contains the Arens' space, where $[\underset{\sim}{\mathbf{D}}_2] \neq \underset{\sim}{\mathbf{D}}_2$. □

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Open question

Is it an equivalence?

Symbolic complexity

Hardness

CoPolish spaces

Spaces of open sets

Spaces of open sets

Admissibly represented spaces have interesting categorical properties (cartesian closed).
In particular, if (X, δ) is admissibly represented then so is $\mathcal{O}(X)$, the space of open sets, with the Scott topology.

We now study symbolic and topological complexity on $\mathcal{O}(X)$.

Spaces of open sets

Theorem

Let X be admissibly represented. On $\mathcal{O}(X)$,

$$[\utilde{\mathbf{D}}_n] = \utilde{\mathbf{D}}_n.$$

$$\begin{array}{ccc}
[\utilde{\mathbf{D}}_\omega] & \longleftarrow\!\rhook & \utilde{\mathbf{D}}_\omega \\
\vdots & & \vdots \\
[\utilde{\mathbf{D}}_n] & = & \utilde{\mathbf{D}}_n \\
\vdots & & \vdots \\
[\utilde{\mathbf{D}}_2] & = & \utilde{\mathbf{D}}_2 \\
[\utilde{\mathbf{\Sigma}}^0_1] & = & \utilde{\mathbf{\Sigma}}^0_1
\end{array}$$

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Question

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The answer depends on the compactness properties of X.

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What about higher complexity levels on $\mathcal{O}(X)$?

We restrict ourselves to the case when X is Polish.

Answer

The answer depends on the compactness properties of X.

Best case: X is locally compact, e.g. $X = \mathbb{R}$.
Worst case: X is not σ-compact, e.g. $X = \mathbb{N}^{\mathbb{N}}$ (Baire space).

Open subsets of Polish spaces

Let $X_{\text{nk}} = \{x \in X : x \text{ has no compact neighborhood}\}$.

Polish spaces are divided in 4 classes

Class I: $X_{\text{nk}} = \emptyset$ (i.e., X is locally compact),

Class II: $X_{\text{nk}} \neq \emptyset$ is finite,

Class III: X_{nk} is infinite and X is σ-compact,

Class IV: X is not σ-compact.

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Class IV: X is not σ-compact. Ex: $\mathbb{N}^{\mathbb{N}}$

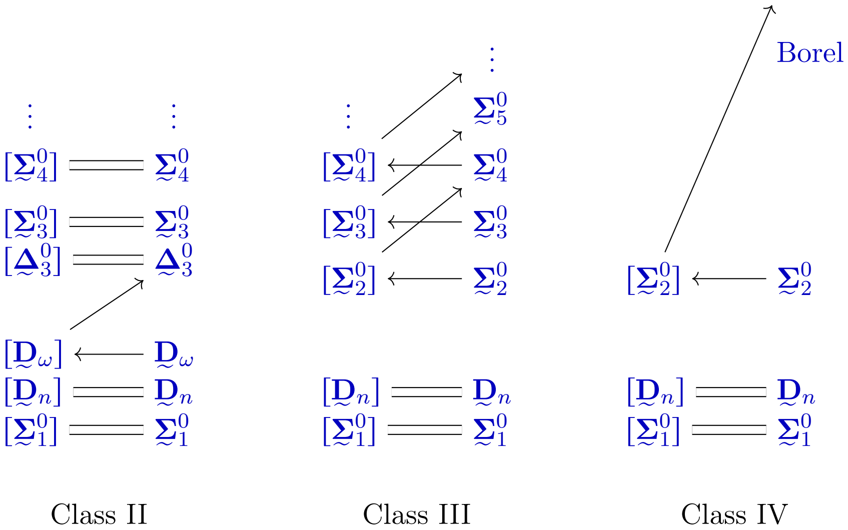
Open subsets of Polish spaces

If $X \in$ Class I then $\mathcal{O}(X)$ is countably-based, so (de Brecht 13) on $\mathcal{O}(X)$,

$$\begin{array}{ccc} \vdots & & \vdots \\ [\underset{\sim}{\mathbf{\Sigma}}^0_3] & = & \underset{\sim}{\mathbf{\Sigma}}^0_3 \\ [\underset{\sim}{\mathbf{\Sigma}}^0_2] & = & \underset{\sim}{\mathbf{\Sigma}}^0_2 \\ [\underset{\sim}{\mathbf{\Delta}}^0_2] & = & \underset{\sim}{\mathbf{\Delta}}^0_2 \\ [\underset{\sim}{\mathbf{D}}_\alpha] & = & \underset{\sim}{\mathbf{D}}_\alpha \\ [\underset{\sim}{\mathbf{\Sigma}}^0_1] & = & \underset{\sim}{\mathbf{\Sigma}}^0_1 \end{array}$$

Symbolic complexity $\equiv$ Topological complexity.

Open subsets of Polish spaces



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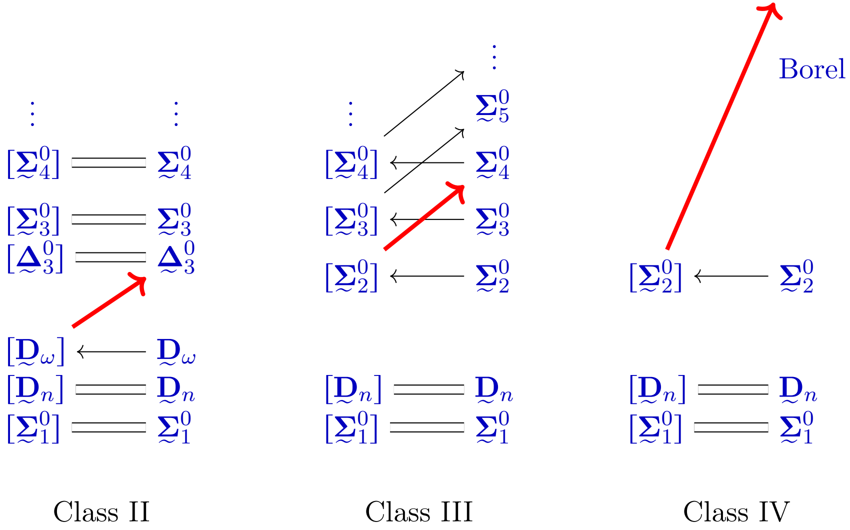
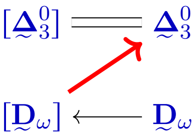


Open subsets of Polish spaces

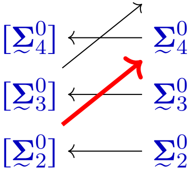
Some $A \in [\boldsymbol{D}_\omega]$ is $\boldsymbol{\Delta}^0_3$-complete*.

- $[\boldsymbol{\Sigma}^0_2] \subseteq \boldsymbol{\Sigma}^0_4$ and
- some $A \in [\boldsymbol{\Sigma}^0_2]$ is $\boldsymbol{\Sigma}^0_3$-complete*.

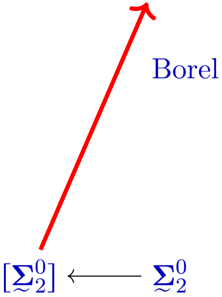
Some $A \in [\boldsymbol{\Sigma}^0_2]$ is not Borel.



Class II



Class III



Class IV

Open subsets of Polish spaces

Open question

For $\mathbf{X} \in$ Class III, is there a $\utilde{\mathbf{\Sigma}}^0_4$-complete* set in $[\utilde{\mathbf{\Sigma}}^0_2]$?
or $[\utilde{\mathbf{\Sigma}}^0_2] \subseteq \utilde{\mathbf{\Sigma}}^0_3$?

Open subsets of the Baire space

Building a set $A \in [\Sigma_2^0]$ which is not Borel.

Proof sketch

- $\mathcal{N}$ is not locally compact,

[1] Escardo, Heckmann. Topologies on Spaces of Continuous Functions (2001)

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  - Which is witnessed by the set

$$\mathscr{E} = \{(f, U) \in \mathcal{N} \times \mathcal{O}(\mathcal{N}) : f \in U\},$$

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  - And $\mathscr{E}$ is not even Borel in the product topology.

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  - Which is witnessed by the set
    $$\mathscr{E} = \{(f, U) \in \mathcal{N} \times \mathcal{O}(\mathcal{N}) : f \in U\},$$
  - And $\mathscr{E}$ is not even Borel in the product topology.
- It is possible to "embed" $\mathscr{E}$ in $\mathcal{O}(\mathcal{N})$ as a $[\Sigma_2^0]$-subset.

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Conclusion

- Several results from DST extend to represented spaces, using symbolic rather than topological complexity.

  Hausdorff-Kuratowski theorem, Wadge lemma
- Is it possible to better understand symbolic complexity classes?

  How to describe $\underset{\sim}{\mathbf{D}}_2$-subsets of $\mathbb{R}[X]$?
- Do the current notions of topological complexity make sense when the space is not countably-based?