Characterising Continuously Realizable Multifunctions

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Survey

- Multivalued functions
- The Characterisation Theorem by Brattka and Hertling
- The powerspace $\mathcal{K}_{+\mathrm{b}}(\mathsf{Y})$
- A more general Characterisation Theorem

Multifunctions

Remember

The following maps are incomputable:

- Tests like $x < y$, $x \geq y$ etc.
- Zero-finding for polynomials:
  - There is no computable selection function $s\colon \mathbb{R}^3 \to \mathbb{R}$ s.t. $s(a,b,c)$ is a zero of $x^3 + ax^2 + bx + c$.

Alternative

- Replace $x < y$ by the *finite precision test* $(x <_\varepsilon y)$:

$$(x <_\varepsilon y) := \begin{cases} \{\mathtt{true}\} & \text{if } x \leq y - \varepsilon \\ \{\mathtt{true}, \mathtt{false}\} & \text{if } y - \varepsilon < x < y \\ \{\mathtt{false}\} & \text{if } x \geq y \end{cases}$$

- Compute the relation $\mathrm{Z} \subseteq \mathbb{R}^3 \times \mathbb{R}$ given by

$$(a,b,c)\,\mathrm{Z}\,x \quad \text{iff} \quad x^3 + ax^2 + bx + c = 0$$

Definition

- A *multifunction* *(or computational task)* F is a relation between represented spaces X and Y, written as $F\colon \mathsf{X} \rightrightarrows \mathsf{Y}$.
- X is the *input space*, Y is the *output space* of F.
- Notation: $F[x] := \{y \in \mathsf{Y} \mid (x, y) \in F\}$.
- $H\colon \mathsf{X} \rightrightarrows \mathsf{Y}$ is a *tightening* for F, if $\forall x \in \mathsf{X}.\, \emptyset \neq H[x] \subseteq F[x]$.

Remark

We will assume every multifunction to be *total*, i.e. $F[x] \neq \emptyset$ for all $x \in \mathsf{X}$.

Remember

A *represented space* X is a set endowed with a representation $\delta_{\mathsf{X}}\colon \mathbb{N}^{\mathbb{N}} \dashrightarrow \mathsf{X}$.

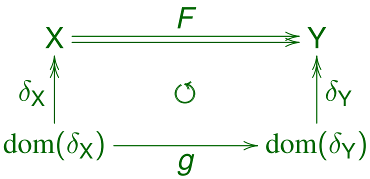
Definition

Let $F\colon \mathsf{X} \rightrightarrows \mathsf{Y}$ be a total multifunction between represented spaces X, Y.

- $g\colon \mathbb{N}^{\mathbb{N}} \dashrightarrow \mathbb{N}^{\mathbb{N}}$ is a *realizer* for F, if

$$\delta_{\mathsf{Y}} g(p) \in F[\delta_{\mathsf{X}}(p)] \qquad \text{for all } p \in \mathrm{dom}(\delta_{\mathsf{X}}).$$

- Diagram:



- F is called *computable (continuously realizable)*, if F has a computable (continuous) realizer g.

Observation

Any realizer g for a tightening H for F is a realizer for F.

Proof

Since $\delta_Y g(p) \in H[\delta_X(p)] \subseteq F[\delta_X(p)]$.

Lemma

- Computably equivalent representations induce the same notion of computable multifunctions.
- Topologically equivalent representations induce the same notion of continuously realizable multifunctions.

Example (Computable multifunctions)

- The finite precision test $(x <_\varepsilon y) \colon \mathbb{R} \times \mathbb{R}_{>0} \times \mathbb{R} \rightrightarrows \mathrm{Bool}$,

$$(x <_\varepsilon y) \ni \begin{cases} \texttt{true} & \text{if } x < y \\ \texttt{false} & \text{if } x > y - \varepsilon \end{cases}$$

- Zero-finding $Z_3 \colon \mathbb{R}^3 \rightrightarrows \mathbb{R}$ of polynomials of degree 3,

$$Z_3[a, b, c] := \{x \in \mathbb{R} \mid x^3 + ax^2 + bx + c = 0\}$$

- The inverse $\delta_X^{-1} \colon X \rightrightarrows \mathrm{dom}(\delta_X)$ of the representation δ_X,

$$\delta_X^{-1}[x] := \{p \in \mathbb{N}^\mathbb{N} \mid \delta_X(p) = x\}$$

Main Question

Find a characterisation of continuously realizable multifunctions in terms of "mathematical" continuity.

Remember:

Theorem

Let X, Y be *admissibly* represented spaces.
Then a function $f\colon X \to Y$ has a continuous realizer iff f is sequentially continuous.

Multifunctions with compact images

Definition

A total multifunction $H\colon \mathsf{X} \rightrightarrows \mathsf{Y}$ has *compact images*, if $H[x]$ is compact in Y for all $x \in \mathsf{X}$.

Proposition

Let X be a computable metric space.
Any computable $F\colon \mathsf{X} \rightrightarrows \mathsf{Y}$ has a computable tightening $H\colon \mathsf{X} \rightrightarrows \mathsf{Y}$ with compact images.

The proof is based on:

Proposition

Any computable metric space X has a representation with compact fibers which is computably equivalent to its Cauchy representation.

Observation

A total multifunction $H\colon X \rightrightarrows Y$ with compact images can be identified with the function $h\colon X \to \mathrm{K}(Y)$, $x \mapsto H[x]$.

Definition

- $\mathrm{K}(Y) := \{$all non-empty compact subsets of $Y\}$
- Basic sets of the *lower Vietoris topology* on $\mathrm{K}(Y)$:

$$\Diamond V := \{K \text{ compact} \mid K \cap V \neq \emptyset\}$$

  where V is open in Y.
- $h\colon X \to \mathrm{K}(Y)$ is *lower semi-continuous*, if h is continuous w.r.t. the lower Vietoris topology.

Characterisation Theorem (V. Brattka & P. Hertling 1994)
Let X, Y be separable metric spaces.
Then $F\colon X \rightrightarrows Y$ has a continuous realizer if, and only if, F has a lower semi-continuous tightening $h\colon X \to \mathrm{K}(Y)$.

Proposition

- Let Y be a QCB_2-space without a countable base.
- Let κ_+ be the positive representation for $\mathrm{K}(\mathsf{Y})$.

Then

- κ_+ is lower semi-continuous.
- But viewed as a multifunction $\kappa_+ : \mathrm{dom}(\kappa_+) \rightrightarrows \mathsf{Y}$, κ_+ has no continuous realizer.

Remark

- This means that compact choice for Y is not continuously realizable w.r.t. κ_+.
- *Compact choice*: selecting an element in a compact set.
- By contrast, compact choice for any computable metric space M is computable w.r.t. its κ_+.

The powerspace $\mathcal{K}_{+\mathrm{b}}$

Aim

Construct a more informative representation for $\mathrm{K}(\mathsf{Y})$ admitting continuous realizability of compact choice.

Definition

Let Y be an admissibly represented QCB_2-space.

- Define a representation $\kappa_{+\mathrm{b}}$ for $\mathrm{K}(\mathsf{Y})$ by

$$\kappa_{+\mathrm{b}}\langle p, b\rangle = K \quad \text{iff} \quad \kappa_{+}(p) = K \;\&\; K \subseteq \kappa_{-}(b)$$

  where κ_{+}, κ_{-} are the canonical positive / negative representations for $\mathrm{K}(\mathsf{Y})$ used in Computable Analysis.
- Define $\mathcal{K}_{+\mathrm{b}}(\mathsf{Y}) := \big(\mathrm{K}(\mathsf{Y}), \kappa_{+\mathrm{b}}\big)$.
- Define $\mathcal{K}_{+}(\mathsf{Y}) := \big(\mathrm{K}(\mathsf{Y}), \kappa_{+}\big)$.

Lemma (The convergence relation)
A sequence $(K_n)_n$ converges to K_∞ in $\mathcal{K}_{+b}(\mathsf{Y})$ iff:

(a) $(K_n)_n$ converges to K_∞ w.r.t. the lower Vietoris topology &

(b) $\bigcup_{n\in\mathbb{N}} K_n$ is relatively compact.

Example

- $\big([0,n]\big)_n$ converges to $[0,1]$ in $\mathcal{K}_{+}(\mathbb{R})$,
- but it is divergent in $\mathcal{K}_{+b}(\mathbb{R})$.
- $\big([0,1-2^{-n}]\big)_n$ converges to $[0,1]$ in $\mathcal{K}_{+b}(\mathbb{R})$,
- but not to $[0,2]$.

Quasi-normal spaces (CCA 2008)

- Quasi-normal space $=$ a QCB-space that arises as the sequentialisation of a normal space.
- Examples:
  - All separable metric spaces
  - The space $\mathcal{D}$ of distributions
  - The Kleene-Kreisel spaces $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}, \mathbb{N}^{\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}}, \ldots$
  - All Co-Polish spaces
- Quasi-normal spaces have excellent closure properties:
  - cartesian closed
  - countable products and equalisers ($\approx$ subspaces)
  - countable co-products and co-equalisers
- Quasi-normal spaces comprise many Hausdorff spaces used in Computable Functional Analysis.

Proposition

Let Y be quasi-normal.

- $\mathcal{K}_{+\mathrm{b}}(\mathsf{Y})$ is an admissibly represented limit space.
- $\mathcal{K}_{+\mathrm{b}}(\mathsf{Y})$ is not topological, unless Y is compact.
- Compact choice for Y is continuously realizable w.r.t. $\kappa_{+\mathrm{b}}$ (i.e. there is a continuous selector $S\colon \mathrm{dom}(\kappa_{+\mathrm{b}}) \to \mathsf{Y}$ such that $S(p) \in \kappa_{+\mathrm{b}}(p)$).

Proposition

Let X have a countable base, let Y be quasi-normal.

Every continuous function $f\colon \mathsf{X} \to \mathcal{K}_{+\mathrm{b}}(\mathsf{Y})$ can be 'blown up' to a upper semi-continuous function $\overline{f}\colon \mathsf{X} \to \mathcal{K}_{-}(\mathsf{Y})$ such that $\forall x \in \mathsf{X}.\, f(x) \subseteq \overline{f}(x)$.

Characterisation Theorem

Let X be separable metrisable, let Y be quasi-normal.

Then $F\colon \mathsf{X} \rightrightarrows \mathsf{Y}$ has a continuous realizer if, and only if, F has a sequentially continuous tightening $h\colon \mathsf{X} \to \mathcal{K}_{+\mathrm{b}}(\mathsf{Y})$.

Remark

- "$\Longleftarrow$" holds for any represented space X.
- "$\Longrightarrow$" holds for any admissibly represented QCB_2-space Y.

Question

Can we generalise this to input spaces $\mathsf{X} \notin \omega\mathsf{Top}$?

Counterexample

Let Poly be the Co-Polish space of polynomials.
Define $\mathrm{DegreeBound}\colon \mathsf{Poly} \rightrightarrows \mathbb{N}$ by

$$\mathrm{DegreeBound}[P] \ni k \;:\Longleftrightarrow\; k \geq \mathrm{Degree}(P).$$

Then

- DegreeBound is computable.
- But DegreeBound does not have a continuously realizable tightening with compact images.

Lemma

For every non-metrisable Co-Polish space X there is

- a computable $F\colon \mathsf{X} \rightrightarrows \mathbb{N}$
- that does not have a continuously realizable tightening with compact images.

Remember

A *Co-Polish space* is the direct limit of an increasing sequence of compact metrisable subspaces X_k.

Lemma

For every $\mathsf{X} \in \mathsf{QCB}_1 \setminus \omega\mathsf{Top}$ there is

- a continuously realizable $F\colon \mathsf{X} \rightrightarrows \mathbb{N}$
- that does not have a lower semi-continuous tightening $h\colon \mathsf{X} \to \mathrm{K}(\mathbb{N})$.

Simple Fact

The inverse $\delta_X^{-1} : X \rightrightarrows \mathrm{dom}(\delta_X)$ is computable.

Proposition

Let X be an admissibly represented QCB_2-space. TFAE:

(a) δ_X^{-1} has a continuous tightening $h : X \to \mathcal{K}_{+b}(\mathrm{dom}(\delta_X))$.

(b) X is separable metrisable.

Corollary

The ‘only-if-part’ of the Characterisation Theorem holds for metrisable input spaces X *only*.

Summary

- Multivalued functions form an indispensable tool in Computable Analysis.
- A computable multifunctions on a computable metric space X has a computable tightening with compact images.
- The powerspace $\mathcal{K}_{+\mathrm{b}}(\mathsf{Y})$ has nice properties (at least) for quasi-normal spaces.
- $\mathcal{K}_{+\mathrm{b}}(\mathsf{Y})$ allows us to characterise continuously realizable multifunctions from metrisable spaces to quasi-normal spaces.
- Open problem:

  Find a characterisation of continuously realizable multifunctions on non-metrisable input spaces.