On Computability over Metric Bundles

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Seventeenth International Conference on Computability and Complexity in Analysis

September 10th, 2020

Facts + Aim

Facts

- The category of bundles of metric spaces over a topological space T is a generalization of the category of metric spaces.
- There exists an object, in category of metric bundles, that could be thought as the counterpart of the *real numbers object* in the category fo metric spaces.

Aim

- To explore the effectivity content in the category of metric bundles in the framework of TTE.
- Introduce, develop and investigate TTE-style computability on metric bundles over a topological space T.

Introduction

Results:

- A first non-trivial relationship between the Category of Computable Topological Spaces and the Category of Metric Bundles is established.
- The concept of bundle of computable metric spaces is introduced.
- The effects of imposing computability conditions on the base space in a metric bundle are studied.
- An Existence Theorem of Bundles of T_0 effective metric bundles is established.
- General conditions to obtain a computable structure in a metric bundle are established.

Preliminars

Let G and T be non-empty sets. Let $p : G \longrightarrow T$ be a surjective function.

Definition. (Selections and Sections)

- A selection for p is a function $\alpha : Q \longrightarrow G$, with $Q \subseteq T$, such that $p \circ \alpha$ is the identity map on Q. If $Q = T$, α is a global selection.
- If T is a topological space and Q is an open subset of T, α is a local selection.
- If G and T are topological spaces, a continuous selection is called a section for p.

Preliminars

Definition. (Metric for p and ϵ-tubes)

A function $p : G \times G \longrightarrow [0, +\infty]$ is a metric for p iff for all $u, v, w \in G$,

- $p(u) \neq p(v) \iff d(u, v) = +\infty$,
- $d(u, v) = 0 \iff u = v$,
- $d(u, v) = d(v, u)$, and
- $d(u, v) \leq d(u, w) + d(w, v)$.

The set $\mathcal{T}_\epsilon(\alpha) = \{u \in G \mid p(u) \in dom(\alpha) \wedge d(u, \alpha(p(u))) < \epsilon\}$ is the ϵ-tube around α.

Preliminars

Definition. (Bundles of metric spaces)

Let G and T be topological spaces, let $p : G \longrightarrow T$ be a continuous surjective function, and let d a metric for p, such that for every $u \in G$ and every $\epsilon > 0$, there is a local selection α such that $u \in \mathcal{T}_\epsilon(\alpha)$.

Then (G, p, T) is a bundle of metric spaces, provided that the collection of sets $\mathcal{T}_\epsilon(\alpha)$, where $\epsilon > 0$ and α runs throughout the local selections for p, is a base for the topology of G.

Preliminars

Theorem of Existence of Metric Bundles (1)

Let T a topological space, G be a non-empty set, $p : G \longrightarrow T$ a surjective function, d be a metric for p, and Γ a family of local selections for p.

If

- For every $u \in G$ and every $\epsilon > 0$, there exists $\alpha \in \Gamma$ such that $u \in \mathcal{T}_\epsilon(\alpha)$.
- For every $(\alpha, \beta) \in \Gamma \times \Gamma$, the function $\Phi_{\alpha\beta} : dom(\alpha) \cap dom(\beta) \longrightarrow \overline{\mathbb{R}}$, defined by $\Phi_{\alpha\beta}(t) = d(\alpha(t), \beta(t))$, is upper semicontinuous.

Then...

Preliminars

Theorem of Existence of Metric Bundles (2)

Then G can be equipped with a topology $\mathfrak{T}$ such that

- If $\mathcal{B} = \{\mathcal{T}_\epsilon(\alpha_Q)\}$, where $\epsilon > 0$, Q is an open subset of $dom(\alpha)$, $\alpha \in \Gamma$, then $\mathcal{B}$ is a base for $\mathfrak{T}$.
- Every $\alpha \in \Gamma$ is a section.
- (G, p, T) is a bundle of metric spaces.

Computable Topological Spaces as Metric Bundles

- Every computable topological space $\mathfrak{X} = (X, \sigma, \nu)$ induces the existence of a metric bundle where the fiber space is the domain of the representation $\delta_{\mathfrak{X}}$.
- The metric bundle associated with $\mathfrak{X}$ contains, as sections, the transformation functions between information blocks, capturing the possibility of construct a name for an object from the name of another object.

E.T.C. Th.ECEM

Bundles of Computable Metric Spaces

Definition (Bundles of Computable Metric Spaces - BCMS)

If G and T are topological spaces, $p : G \longrightarrow T$ is a continuous surjective function, and d is a metric for p, then (G, p, T) is a Bundle of Computable Metric Spaces if and only if

- For all $u \in G$ and for all $\epsilon > 0$ there exists a local section α for p such that $u \in \mathcal{T}_\epsilon(\alpha)$.
- The family $\{\mathcal{T}_\epsilon(\alpha)\}$, with $\epsilon > 0$ and α local section for p, is a base for the topology of G.
- For all $t \in T$, there exists $Q_t \subseteq p^{-1}(t)$, d_t and ν_t such that $(p^{-1}(t), d_t, Q_t, \nu_t)$ is a computable metric space.

E.M.C.

Bundles of Computable Metric Spaces

Examples (Bundles of Computable Metric Spaces - BCMS)

- Bundle of countinuous functions over a topological space T with values in a computable metric space $\mathfrak{E} = (E, d, Q, \nu)$.
- Sheaf of germs of rational polygons over a recursive interval.

Ej. 1 Ej. 2

Metric Bundles with Computable Base

Let

- $\mathfrak{X} = (X, \tau, \beta, \nu)$ be a T_0 computable space.
- G be a set with a cardinality at most that of the continuum.
- $p : G \longrightarrow X$ be a surjective function.
- d be a metric for p.
- Γ be a family of local selections for p, such that for all $u \in G$ and for all $n \in \mathbb{N}$ there exists $\alpha \in \Gamma$ with $u \in \mathcal{T}_{2^{-n}}(\alpha)$.
- $\nu_\Gamma :\subseteq \Sigma^* \longrightarrow \Gamma$ be a notation for Γ.

Metric Bundles with Computable Base

Definition (Cauchy representation of G according to Γ)

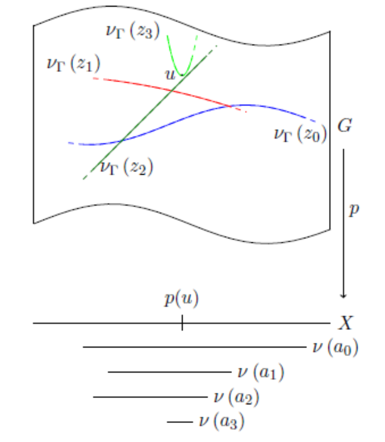
q is a δ_{Γ^C}-name for u if $q = \langle a, b \rangle$,

where

a indictes the fiber $G_{\delta_{\mathfrak{X}}(a)}$ where u is located (a is a $\delta_{\mathfrak{X}}$-name for $p(u)$), and

b is an ordered list of ν_Γ-names for selections ζ_n ($\zeta_n := \nu_\Gamma(z_n)$) in Γ such that the sequence $\{\zeta_n(p(u))\}_{n\in\mathbb{N}}$ converges *rapidly* to u in $G_{p(u)} = G_{\delta_{\mathfrak{X}}(a)}$.

Metric Bundles with Computable Base



Metric Bundles with Computable Base

Definition (Cauchy representation of G according to Γ)

Let $\delta_{\Gamma C} :\subseteq \Sigma^{\omega} \longrightarrow G$ be the representation defined by

$$\delta_{\Gamma C}(q) = u$$

if and only if

- $a \in dom\left(\delta_{\mathfrak{X}}\right)$,
- $q = \langle a, \iota\left(z_0\right) \iota\left(z_1\right) \ldots \rangle$,
- $\delta_{\mathfrak{X}}(a) = p(u)$, and
- $d\left(u, \nu_{\Gamma}\left(z_n\right)\left(p\left(u\right)\right)\right) \leq 2^{-n}$ for all $n \in \mathbb{N}$.

Metric Bundles with Computable Base

Theorem

Let $x \in X$ and $G_x = p^{-1}(x)$. If d is $(\delta_{\Gamma C}, \delta_{\Gamma C}, \rho)$-computable, then there exists d_x, Q_x and ν_x such that $\mathfrak{G}_x = (G_x, d_x, Q_x, \nu_x)$ is a computable metric space.

Metric Bundles with Computable Base

Proof.

Let

- $G_x = p^{-1}(x)$.
- $d_x = d \restriction_{G_x \times G_x}$.
- $Q_x = \{\gamma(x) \mid x \in dom\,(\gamma) \wedge \gamma \in \Gamma\}$.
- $\nu_x :\subseteq \Sigma^* \longrightarrow Q_x$ defined by

$$\nu_x(q) = u \Longleftrightarrow \nu_\Gamma(q)(x) = u.$$

Metric Bundles with Computable Base

Proof. (cont.)

- Q_x is a dense subset of G_x: Let $u \in G_x$. There exists $\zeta \in \Gamma$ such that $d(u, \zeta(p(u))) < \frac{1}{2^n}$, then $\zeta(x) \in Q_x \cap B_x\left(u, \frac{1}{2^n}\right) \neq \emptyset$.
- d_x es (ν_x, ν_x, ρ)-computable: If $\delta_{\mathfrak{X}}(q) = x$, M_d is the Type-2 Turing machine that computes d, and $q_1, q_2 \in dom\,(\nu_x)$, then $\tilde{q}_k = \langle q, \iota(q_k)\iota(q_k)\ldots\rangle$ are such that $\nu_x\,(q_k) = \delta_{\Gamma C}\,(\tilde{q}_k)$ and $d_x\,(q_1, q_2) = d\,(\tilde{q}_1, \tilde{q}_2)$.

Dem.

Metric Bundles with Computable Base

Theorem.

If $\nu_{\mathbb{N}}$ is a standard notation for $\mathbb{N}$, then, there exists a $(\delta_{\Gamma C}, \nu_{\mathbb{N}}, \nu_{\Gamma})$-computable function such that, if $u \in G$ and $n \in \mathbb{N}$ are given, then the function finds a selection $\gamma \in \Gamma$ such that $u \in \mathcal{T}_{2^{-n}}(\gamma)$.

C.E.M.

Metric Bundles with Computable Base

Proof.

Let

$$SEL :\subseteq \Sigma^{\omega} \times \Sigma^{*} \longrightarrow \Sigma^{*}$$

be the function defined by

$$dom(SEL) = dom\left(\delta_{\Gamma C}\right) \times dom\left(\nu_{\mathbb{N}}\right), \text{ and}$$

$$\delta_{\Gamma C}(q) \in \mathcal{T}_{2^{-\nu_{\mathbb{N}}(r)}}(SEL(q,r)).$$

$SEL(q,r)$ is the $\nu_{\mathbb{N}}(r)$-th projection evaluated in the second projection of q, and

$$d\left(\delta_{\Gamma C}(q), \nu_{\Gamma}(SEL(q,r))\left(p\left(\delta_{\Gamma C}(q)\right)\right)\right) < 2^{-n}.$$

Metric Bundles with Computable Base

Theorem (Existence Theorem of T_0 effective metric bundles)

If $\mathfrak{X} = (X, \tau, \beta, \nu)$ is a T_0 effective topological space, G is a non-empty set, $p : G \longrightarrow X$ is a surjective function, d is a metric for p, Γ is a countable family of local selections for p, and $\nu_\Gamma :\subseteq \Sigma^* \longrightarrow \Gamma$ is a notation for Γ, are such that

- For all $u \in G$ and $n \in \mathbb{N}$, there exists a local selection $\gamma \in \Gamma$ such that $u \in \mathcal{T}_{2^{-n}}(\gamma)$, and
- For each pair of local selections $\gamma, \zeta \in \Gamma$, the function $\Phi_{\gamma\zeta} : dom(\gamma) \cap dom(\zeta) \longrightarrow \mathbb{R}$, defined by $\Phi_{\gamma\zeta}(t) = d(\gamma(t), \zeta(t))$, is a upper semicontinuous function,

Then...

Metric Bundles with Computable Base

Theorem (Existence Theorem of T_0 effective metric bundles)

Then G can be equipped with a topology τ_G such that

- The family $\beta_G = \{\mathcal{T}_{2^{-n}}(\gamma_Q)\}$ is a base for the topology τ_G.
- If $\gamma \in \Gamma$ then γ is a section.
- (G, p, X) is a metric bundle.
- There exists a notation $\nu_G :\subseteq \Sigma^* \longrightarrow \beta_G$ such that $\mathfrak{G} = (G, \tau_G, \beta_G, \nu_G)$ is a T_0 effective topological space.
- p is a $(\delta_G, \delta_{\mathfrak{X}})$-computable function.

Metric Bundles with Computable Base

Proof.

1. A direct application of the Existence Theorem of Metric Bundles allows us to conclude the first three items.
2. To define $\nu_G :\subseteq \Sigma^* \longrightarrow \beta_G$ such that $\mathfrak{G} = (G, \tau_G, \beta_G, \nu_G)$ is a T_0 effective topological space.
3. Starting with a δ_G-name for u, construct a $\delta_{\mathfrak{X}}$-name for $p(u)$.

Metric Bundles with Computable Base

To define $\nu_G :\subseteq \Sigma^* \longrightarrow \beta_G$ such that $\mathfrak{G} = (G, \tau_G, \beta_G, \nu_G)$ is a T_0 effective topological space.

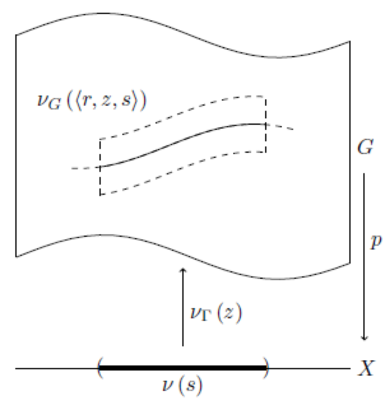
Proof. (definition of ν_G)

Let $\nu_G :\subseteq \Sigma^* \longrightarrow \beta_G$ be the notation defined by

$\nu_G(q) = \mathcal{T}_{2^{-n}}(\gamma_Q) :\Longleftrightarrow q = \langle r, z, s\rangle$ where $\nu_{\mathbb{N}}(r) = n$, $\nu_\Gamma(z) = \gamma$ and $\nu(s) = Q$.

r names the radius 2^{-n} of the tube, z names the section γ, and s names the open subset Q.

Metric Bundles with Computable Base



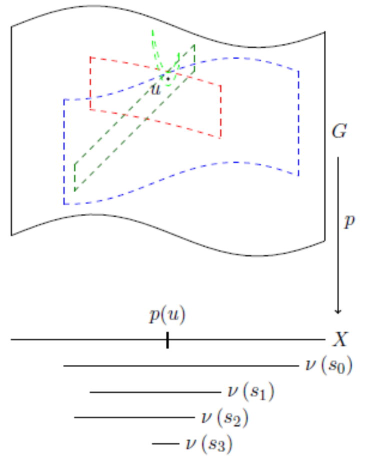
Metric Bundles with Computable Base

Starting with a δ_G-name for u, construct a $\delta_{\mathfrak{X}}$-name for $p(u)$.

Proof. (p is a $(\delta_G, \delta_{\mathfrak{X}})$-computable function)

1. If $\delta_G(q) = u$ then q is a list of tubes containing u.
2. If $\iota(q_k) \lhd q$ then $q_k = \langle r_k, z_k, s_k \rangle$.
3. $\iota(s_0)\,\iota(s_1)\ldots$ is a $\delta_{\mathfrak{X}}$-name for $p(u)$.

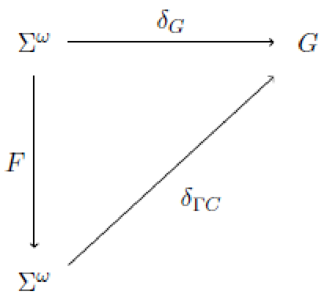
Metric Bundles with Computable Base



Metric Bundles with Computable Base

Theorem

$\delta_G \leq \delta_{\Gamma C}$.



Metric Bundles with Computable Base

Theorem

$\delta_G \leq \delta_{\Gamma C}$.

Proof.

A function that translates a δ_G-name for $u \in G$, to a $\delta_{\Gamma C}$-name for u can be constructed:

Since p is a $(\delta_G, \delta_{\mathfrak{X}})$-computable function, take a such that $\delta_{\mathfrak{X}}(a) = p(u)$.

On the other hand, is possible to construct $b = \iota(z_0)\,\iota(z_1)\ldots$ such that the sequence $\{\nu_\Gamma(z_n)\}_{n\in\mathbb{N}}$ converge to u.

Then, since a, b and $\langle a, b\rangle$ can be computed in parallel, the translation function is a computable function.

Metric Bundles with Computable Base

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Metric Bundles with Computable Base

Theorem

$\delta_{\Gamma C}$ is an admissible representation w.r.t. τ_G.

Proof.

1. $\delta_{\Gamma C}$ is unambiguous.
2. $\delta_{\Gamma C}$ is surjective.
3. $\delta_{\Gamma C}$ is continuous.
4. $\delta_{\Gamma C}$ has the universal property.

Metric Bundles with Computable Base

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Metric Bundles with Computable Base

Proof. ($\delta_{\Gamma C}$ *is unambiguous*)

Let $u, v \in G$ and $q = \langle a, b \rangle \in \Sigma^\omega$ be such that $u \neq v$ and $\delta_{\Gamma C}(q) = u$.

- Case $p(u) \neq p(v)$. There exists $s \in dom\,(\nu)$ such that $p(u) \in \nu\,(s)$ and $p(v) \notin \nu\,(s)$. Since $\iota\,(s) \triangleleft a = \pi_1\,(q)$, then $\delta_{\mathfrak{X}}(a) \neq p(v)$ y $\delta_{\Gamma C}(q) \neq v$.
- Case $p(u) = p(v)$. Let $n \in \mathbb{N}$ be such that $2^{-n} < \frac{1}{2} d\,(u, v)$. Since $\delta_{\Gamma C}\,(q) = u$ and $b = \iota\,(z_0)\,\iota\,(z_1) \ldots$ are such that $d\,(u, \nu_\Gamma\,(z_n)\,(p(u))) < 2^{-n}$, then $d\,(v, \nu_\Gamma\,(z_n)\,(p(v))) > 2^{-n}$ and $\delta_{\Gamma C}(q) \neq v$.

Dem.

Metric Bundles with Computable Base

Proof. *($\delta_{\Gamma C}$ is surjective)*

Let $u \in G$.

- There exists $a \in dom\left(\delta_{\mathfrak{X}}\right)$ such that $\delta_{\mathfrak{X}}(a) = p(u)$.
- For all $n \in \mathbb{N}$ there exists $\zeta_n \in \Gamma$ such that $d\left(u, \zeta_n\left(p(u)\right)\right) < 2^{-n}$. Take $b = \iota\left(z_0\right)\iota\left(z_1\right)\ldots$ such that $z_n \in dom\left(\nu_\Gamma\right)$ and $\nu_\Gamma\left(z_n\right) = \zeta_n$.

Then $\delta_{\Gamma C}\left(\langle a, b\rangle\right) = u$.

Dem.

Metric Bundles with Computable Base

Proof. ($\delta_{\Gamma C}$ *is continuous*)

Let $A = \mathcal{T}_{2^{-n}}(\zeta \restriction_V)$ be a basic open in G. $A = \nu_G(\langle r, z, s\rangle)$.

It is necessary to see that $\delta_{\Gamma C}^{-1}(A) = \delta_{\Gamma C}^{-1}(\mathcal{T}_{2^{-n}}(\zeta \restriction_V)) = \delta_{\Gamma C}^{-1}(\nu_G(\langle r, z, s\rangle)) = \delta_{\Gamma C}^{-1}\left(\mathcal{T}_{2^{-\nu_{\mathbb{N}}(r)}}\left(\nu_\Gamma(z) \restriction_{\nu(s)}\right)\right)$ is open.

Let $q = \langle a, b\rangle \in \delta_{\Gamma C}^{-1}(A)$ where $a = \iota(a_0)\,\iota(a_1)\ldots$ and $b = \iota(z_0)\,\iota(z_1)\ldots$.

Dem.

Metric Bundles with Computable Base

Proof. ($\delta_{\Gamma C}$ *is continuous*)

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Let $q = \langle a, b\rangle \in \delta_{\Gamma C}^{-1}(A)$ where $a = \iota(a_0)\,\iota(a_1)\ldots$ and $b = \iota(z_0)\,\iota(z_1)\ldots$.

Dem.

Metric Bundles with Computable Base

Proof. ($\delta_{\Gamma C}$ *is continuous*)

- Let $k_0 \in \mathbb{N}$ such that $k_0 > \nu_{\mathbb{N}}(r)$ and
$$\frac{1}{2^{k_0}} < \frac{1}{4}\left(\frac{1}{2^n} - d\left(\delta_{\Gamma C}(q), \zeta\left(\delta_{\mathfrak{X}}(a)\right)\right)\right).$$
- Let $k_1 \geq k_0$ such that $\nu\left(a_{k_1}\right) \cap dom\left(\zeta\right) \cap V \neq \emptyset$.

  k_1 exists because $\delta_{\mathfrak{X}}(a) \in dom(\zeta)$ and a is a $\delta_{\mathfrak{X}}$-name for $p\left(\delta_{\Gamma C}(q)\right)$.

Dem.

Metric Bundles with Computable Base

Proof. ($\delta_{\Gamma C}$ *is continuous*)

- Let $k_2 \geq k_1$ such that

$$\nu\left(a_{k_2}\right) \subseteq \left\{x \in X \mid d\left(\zeta(x), \nu_\Gamma\left(z_{k_0}\right)(x)\right) < \frac{1}{2^{n+1}}\right\}.$$

k_2 exists because $\Phi_{\nu_\Gamma(z),\nu_\Gamma\left(z_{k_0}\right)}$ is an upper semicontinuous function.

Dem.

Metric Bundles with Computable Base

Proof. *($\delta_{\Gamma C}$ is continuous)*

Let $B = \langle \iota(a_0) \ldots \iota(a_{k_2}), \iota(z_0) \ldots \iota(z_{k_2}) \rangle \Sigma^\omega \cap dom(\delta_{\Gamma C})$.

To see that $B \subseteq \delta_{\Gamma C}^{-1}(A)$ and therefore $\delta_{\Gamma C}^{-1}(A)$ contains a neighborhood of q.

Dem.

Metric Bundles with Computable Base

Proof. ($\delta_{\Gamma C}$ *is continuous*)

Let $B = \langle \iota(a_0) \ldots \iota(a_{k_2}), \iota(z_0) \ldots \iota(z_{k_2}) \rangle \Sigma^\omega \cap dom(\delta_{\Gamma C})$.

To see that $B \subseteq \delta_{\Gamma C}^{-1}(A)$ and therefore $\delta_{\Gamma C}^{-1}(A)$ contains a neighborhood of q.

Dem.

Metric Bundles with Computable Base

Proof. ($\delta_{\Gamma C}$ *is continuous*)

Let $q' = \langle a', b' \rangle \in B$ where $a' = \iota(a'_0)\,\iota(a'_1)\ldots$ and $b' = \iota(z'_0)\,\iota(z'_1)\ldots$.

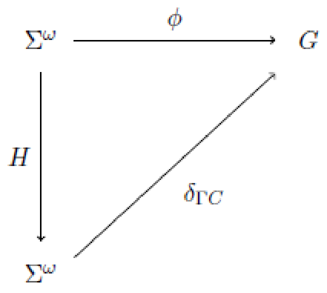
- If $\delta_{\mathfrak{X}}(a) = \delta_{\mathfrak{X}}(a')$ then $\delta_{\Gamma C}(q') \in A$ since $d\left(\delta_{\Gamma C}(q'), \zeta\left(\delta_{\mathfrak{X}}(a')\right)\right) < 2^{-n}$.
- If $\delta_{\mathfrak{X}}(a) \neq \delta_{\mathfrak{X}}(a')$ then $\delta_{\Gamma C}(q') \in A$ since $\delta_{\mathfrak{X}}(a') \in \nu(a_{k_2}) \cap dom(\zeta) \cap V$ and therefore $d\left(\delta_{\Gamma C}(q'), \zeta\left(\delta_{\mathfrak{X}}(a')\right)\right) < \frac{1}{2^n}$.

Then $B \subseteq \delta_{\Gamma C}^{-1}(A)$, $\delta_{\Gamma C}^{-1}(A)$ is open, and $\delta_{\Gamma C}$ is continuous.

Dem.

Metric Bundles with Computable Base

Proof. ($\delta_{\Gamma C}$ *has the universal property)*



Metric Bundles with Computable Base

Proof. *($\delta_{\Gamma C}$ has the universal property)*

Let $\phi :\subseteq \Sigma^\omega \longrightarrow G$ be a continuous function. We are looking for a continuous function $H :\subseteq \Sigma^\omega \longrightarrow \Sigma^\omega$ such that $\phi = \delta_{\Gamma C} \circ H$.

Dem.

Metric Bundles with Computable Base

Proof. ($\delta_{\Gamma C}$ *has the universal property)*

Let η_G and η_X be enumerations of the bases β_G and β.

Let $q \in dom(\phi)$, $u = \phi(q)$, and V an open subset of G with $u \in V$.

Since ϕ and p are continuous functions,

$$\lim_{n\to\infty} \phi\left(q^{<n}\Sigma^{\omega}\right) = \{\phi(q)\}$$

and

$$\lim_{m\to\infty} p\left(\phi\left(q^{<m}\Sigma^{\omega}\right)\right) = \{p(\phi(q))\}.$$

Dem.

Metric Bundles with Computable Base

Proof. ($\delta_{\Gamma C}$ *has the universal property*)

For all $k \in \mathbb{N}$ there exists $n_k, j_k \in \mathbb{N}$ and $(r_{j_k}, z_{j_k}, s_{j_k})$ such that

- $\eta_G(j_k) = \nu_G(\langle r_{j_k}, z_{j_k}, s_{j_k} \rangle)$,
- $\nu_{\mathbb{N}}(r_{j_k}) > k$, and
- $\phi(q) \in \phi(q^{<n_k}\Sigma^\omega) \subseteq \eta_G(j_k) \subseteq V$.

Note that $d(u, \nu_\Gamma(z_{j_k})(p(u))) < 2^{-k}$.

Dem.

Metric Bundles with Computable Base

Proof. ($\delta_{\Gamma C}$ has the universal property)

Let

$$H_b(q) = \iota\,(z_{j_0})\,\iota\,(z_{j_1})\,\ldots.$$

Then the prefix

$$\iota\,(z_{j_0})\,\iota\,(z_{j_1})\ldots\iota\,(z_{j_K})$$

of $H_b(q)$ is uniquely determined by the prefix $q^{<\max\{n_0,\ldots,n_K\}}$ of q.

Dem.

Metric Bundles with Computable Base

Proof. ($\delta_{\Gamma C}$ *has the universal property*)

For all $k \in \mathbb{N}$ there exists $m_k, i_k \in \mathbb{N}$ and a_{i_k} such that

- $\eta_X(i_k) = \nu(a_{i_k})$, and
- $p(\phi(q)) \in p\left(\phi\left(q^{<m_k}\Sigma^\omega\right)\right) \subseteq \eta_X(i_k) \subseteq p(V)$.

Dem.

Metric Bundles with Computable Base

Proof. *($\delta_{\Gamma C}$ has the universal property)*

Let

$$H_a(q) = \iota\left(a_{i_0}\right)\iota\left(a_{i_1}\right)\ldots.$$

Then, the prefix

$$\iota\left(a_{i_0}\right)\iota\left(a_{i_1}\right)\ldots\iota\left(a_{i_K}\right)$$

of $H_a(q)$ is uniquely determined by the prefix $q^{<\max\{m_0,\ldots,m_K\}}$ of q.

Dem.

Metric Bundles with Computable Base

Proof. ($\delta_{\Gamma C}$ *has the universal property*)

Let

$$H(q) = \langle H_a(q), H_b(q) \rangle .$$

H is a continuous function because the prefix $\langle \iota(a_{i_0}) \dots \iota(a_{i_K}), \iota(z_{j_0}) \dots \iota(z_{j_K}) \rangle$ of $H(q)$ is uniquely determined by the prefix $q^{<\max\{M_K, N_K\}}$ of q, where $M_K = \max\{m_0, \dots, m_K\}$ and $N_K = \max\{n_0, \dots, n_K\}$,

and $H(q)$ is a $\delta_{\Gamma C}$-name for $\phi(q)$.

Dem.

Metric Bundles with Computable Base

Proof.

Then

1. $\delta_{\Gamma C}$ is unambiguous.
2. $\delta_{\Gamma C}$ is surjective.
3. $\delta_{\Gamma C}$ is continuous.
4. $\delta_{\Gamma C}$ has the universal property.

and $\delta_{\Gamma C}$ is admissible.

Computable Bundles of Metric Spaces

Theorem (Existence Theorem of Computable Bundles of Metric Spaces)

If $\mathfrak{X} = (X, \tau_{\mathfrak{X}})$ is a T_0 topological space and $\mathcal{B}_X$ contable base for the topology $\tau_{\mathfrak{X}}$, G is a set with cardinality at most that of the continuum, $p : G \longrightarrow X$ a surjective function, d a metric for p, Γ a countable family of local selections for p, $\nu_\Gamma :\subseteq \Sigma^* \longrightarrow \Gamma$ a notation for Γ, are such that

- For all $u \in G$ and $n \in \mathbb{N}$, there exists a local selection $\alpha \in \Gamma$ such that $u \in \mathcal{T}_{2^{-n}}(\alpha)$.
- For all selections $\alpha, \beta \in \Gamma$, the function $\Phi_{\alpha\beta} : dom(\alpha) \cap dom(\beta) \longrightarrow \mathbb{R}$, defined by $\Phi_{\gamma\zeta}(t) = d(\gamma(t), \zeta(t))$, is an upper semicontinuous function.

Computable Bundles of Metric Spaces

Theorem (Existence Theorem of Computable Bundles of Metric Spaces)

Then G can be equipped with a topology $\tau_{\mathfrak{G}}$ such that

- The family $\beta_G = \{\mathcal{T}_{2^{-n}}(\gamma_Q)\}$ is a countable base for the topology $\tau_{\mathfrak{G}}$.
- If $\alpha \in \Gamma$ then α is a section.
- (G, p, X) is a metric bundle.
- There exists a $\tau_{\mathfrak{G}}$-admissible representation for G.

Dem.

Computable Bundles of Metric Spaces

Results:

- A first non-trivial relationship between the Category of Computable Topological Spaces and the Category of Metric Bundles is established.
- The concept of bundle of computable metric spaces is introduced.
- The effects of imposing computability conditions on the base space in a metric bundle are studied.
- An Existence Theorem of Bundles of T_0 effective metric bundles is established.
- General conditions to obtain a computable structure in a metric bundle are established.

On Computability over Metric Bundles

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