The Discontinuity Problem

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Is There a Simplest Natural Unsolvable Problem?

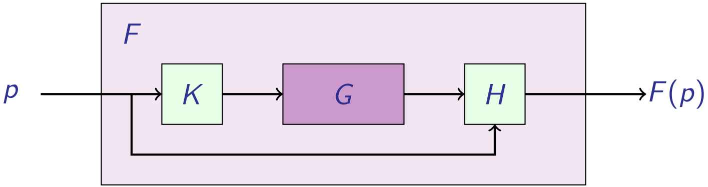
- **Simplicity** can be measured in different ways. For instance, the weakest natural unsolvable problem with respect to Turing reducibility seems to be the halting problem, whereas there are weaker natural problems with respect to many-one-reducibility.

- **Naturality** is supposed to express that the problem is not "artificially constructed" or exists only by invocation of the Axiom of Choice etc. A natural problem should be one with a simple definition that is of independent genuine interest.

- **Solvability** again refers to the underlying reducibility. Here we are interested in problems as multi-valued functions with respect to Weihrauch reducibility and solvability can either be meant in the computable or in the continuous sense.

Weihrauch Reducibility

Let $f :\subseteq X \rightrightarrows Y$ and $g :\subseteq Z \rightrightarrows W$ be two multi-valued functions.



- f is Weihrauch reducible to g, $f \leq_{\mathrm{W}} g$, if there are computable $H, K :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that $H\langle \mathrm{id}, GK\rangle \vdash f$ whenever $G \vdash g$.
- We write $f \leq_{\mathrm{W}}^{*} g$ for the continuous version of Weihrauch reducibility, where the translation functions H, K are only required to be continuous.
- The mentioned reducibilities all induce lattices. The lattice for $\leq_{\mathrm{W}}$ is usually referred to as Weihrauch lattice.

LPO as Simplest Discontinuous Function

By $\text{LPO} : \mathbb{N}^{\mathbb{N}} \to \{0,1\}$ we denote the limited principle of omniscience, which is defined by $\text{LPO}(p) = 1 :\iff p = 000\ldots$.

Theorem (Folklore)

For a function $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ the following are equivalent:

1. $\text{LPO} \leq_{\mathrm{W}}^* f$,
2. *f is discontinuous.*

1. Early proofs of this result are due to von Stein (1989), Weihrauch (1992), B. (1993).
2. Pauly (2010) has generalized this result to arbitrary topological spaces (using a modified reducibility).
3. If one combines his proof with Schröder's characterization of sequential continuity, then the theorem generalizes to functions $f :\subseteq X \to Y$ on admissibly represented spaces X, Y with sequential continuity in place of continuity.

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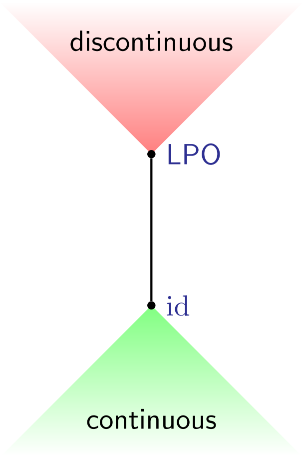
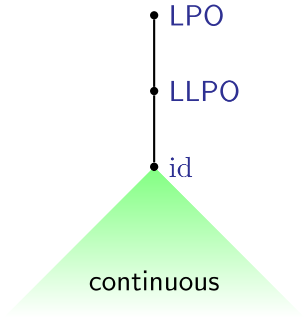
LPO in a Dichotomy



Functions $f : X \to Y$ on admissibly represented spaces with respect to continuous Weihrauch reducibility $\leq_{\mathrm{W}}^{*}$.

The Picture for Multi-Valued Problems



$\mathsf{C}_2 = \mathsf{LLPO} :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \{0,1\}$ is multi-valued.

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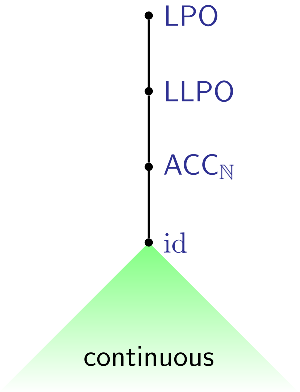

$\mathsf{LLPO}_\infty = \mathsf{ACC}_{\mathbb{N}} :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}$ is multi-valued.

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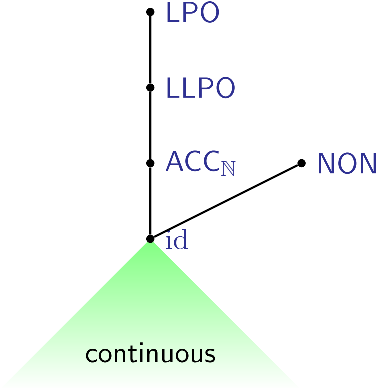
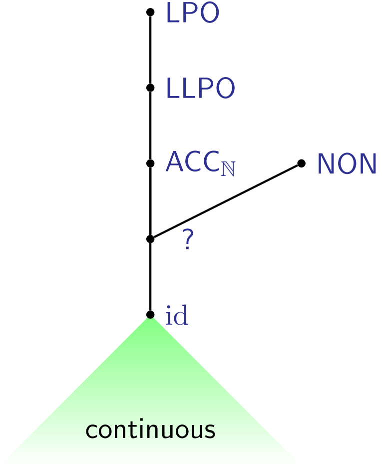


$\mathsf{NON} : \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}, p \mapsto \{q : q \not\leq_{\mathrm{T}} p\}$ is called the non-computability problem.

A Weakest Discontinuous Multi-Valued Problem?



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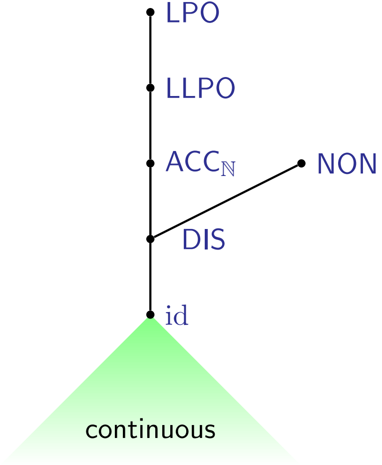

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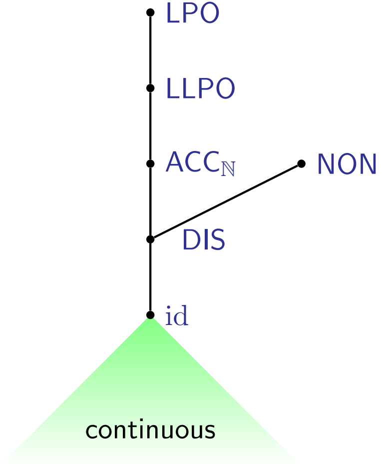

$\mathsf{DIS} : \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}, p \mapsto \{q : \mathsf{U}(p) \neq q\}$, where $\mathsf{U} :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ is a fixed universal computable function.

DIS as Simplest Effectively Discontinuous Problem

Theorem

For a problem $f :\subseteq X \rightrightarrows Y$ the following are equivalent:

1. $\mathrm{DIS} \leq^*_{\mathrm{W}} f$,
2. f *is effectively discontinuous.*

The proof is based on the Recursion Theorem.

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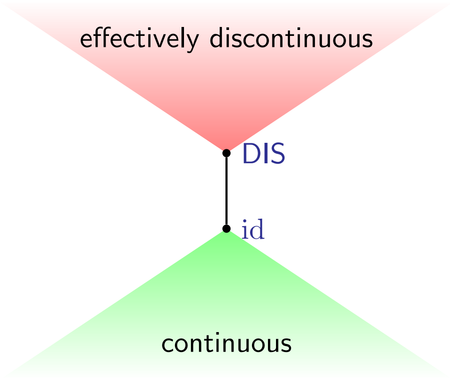

Effective Discontinuity

Let Φ be a defined by $\Phi_q(p) := \mathrm{U}\langle q, p\rangle$.

Definition

Let (X, δ_X) and (Y, δ_Y) be represented spaces. A problem $f :\subseteq X \rightrightarrows Y$ is called effectively discontinuous if there is a continuous $D : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that for all $q \in \mathbb{N}^{\mathbb{N}}$ we obtain

$$D(q) \in \mathrm{dom}(f\delta_X) \text{ and } \delta_Y \Phi_q D(q) \notin f\delta_X D(q).$$

In this case the function D is called a discontinuity function of f.

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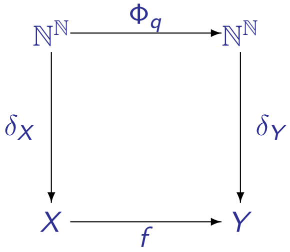

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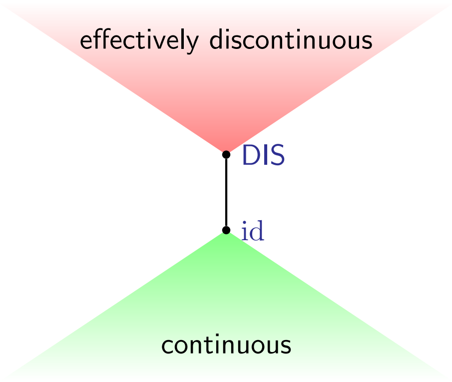

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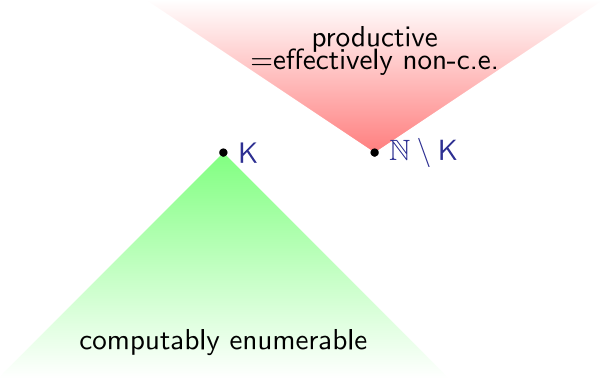

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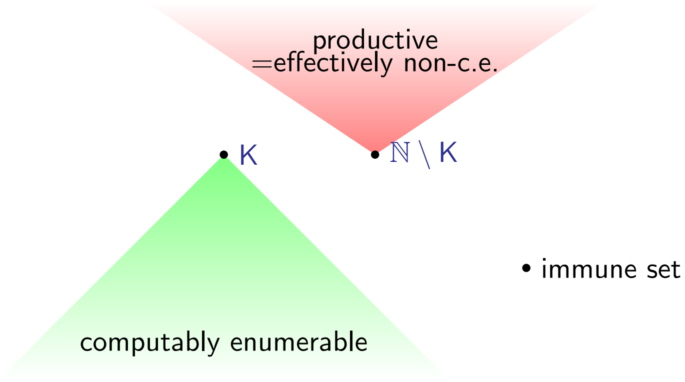

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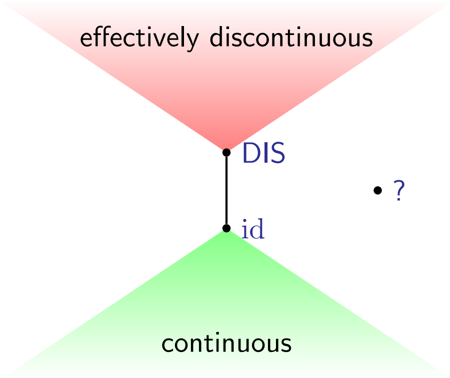

An Unnatural Discontinuous Problem

Theorem

Assuming the Axiom of Choice (AC) there exists a problem $f :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ that is discontinuous, but not effectively so.

1. The fact can be derived from the existence of Bernstein sets (which is a set $B \subseteq \mathbb{N}^{\mathbb{N}}$ such that B as well as its complement have non-empty intersection with every uncountable closed set $A \subseteq \mathbb{N}^{\mathbb{N}}$.)
2. This construction can be seen as an infinitary version of Post's construction of an immune set.
3. By a direct transfinite recursion one can even strengthen the result such that f becomes total and parallelizable.
4. Is the Axiom of Choice (AC) really necessary for this construction?

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The Uniformization Game

We introduce a variant of a Banach-Mazur game.

Definition

Let $f :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ be a problem. Then in a uniformization game f two players I and II consecutively play words

- Player I: $w_0 \quad w_1 \quad w_2 \quad \ldots =: r$,
- Player II: $\quad v_0 \quad v_1 \quad v_2 \quad \ldots =: q$,

with $w_i, v_i \in \mathbb{N}^*$. The concatenated sequences $(r, q) \in (\mathbb{N}^{\mathbb{N}} \cup \mathbb{N}^*)^2$ are called a run of the game f. Player II wins the run (r, q) of f, if $(r, q) \in \mathrm{graph}(f)$ or $r \notin \mathrm{dom}(f)$. Otherwise Player I wins.

Theorem

Consider the game $f :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$. Then the following hold:

1. *f is continuous $\iff$ Player II has a winning strategy for f,*
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The Picture Under the Axiom of Determinacy

Theorem

In $\mathsf{ZF}+\mathsf{DC}+\mathsf{AD}$ *every total* $f: X \rightrightarrows Y$ *on complete separable metric spaces* X *and* Y *has a continuous realizer or is effectively discontinuous (i.e., either* $f \leq^*_{\mathrm{W}} \mathrm{id}$ *or* $\mathsf{DIS} \leq^*_{\mathrm{W}} f$ *holds).*

Proof idea. The theorem can be proved by a reduction of the uniformization game to a Gale-Stewart (Ulam) game. Any such game is determined by the axiom AD, which means that either player I or player II has a winning strategy. □

Corollary

In $\mathsf{ZF}+\mathsf{DC}+\mathsf{AD}$ *every problem* $f :\subseteq X \rightrightarrows Y$ *either satisfies* $f \leq^*_{\mathrm{tW}} \mathrm{id}$ *or* $\mathsf{DIS} \leq^*_{\mathrm{tW}} f$.

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Parallelization and Summation

Parallelization

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For every problem $f :\subseteq X \rightrightarrows Y$ we define its parallelization $\Pi f :\subseteq X^{\mathbb{N}} \rightrightarrows Y^{\mathbb{N}}$ by $\mathrm{dom}(\Pi f) := \mathrm{dom}(f)^{\mathbb{N}}$ and

$$\Pi f(x_n) := \{(y_n) \in Y^{\mathbb{N}} : (\forall n)\ y_n \in f(x_n)\}$$

for all $(x_n) \in X^{\mathbb{N}}$. We usually write $\widehat{f} := \Pi f$ and we call a problem parallelizable if $f \equiv_{\mathrm{W}} \widehat{f}$ holds.

Parallelization is known to be a closure operator on the Weihrauch lattice (and an analogue of the ! operator in linear logic).

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$\widehat{\mathsf{DIS}} \equiv_{\mathrm{W}} \mathsf{NON}$.

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Slogan: Non-computability is the parallelization of discontinuity!

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Here $\overline{Y}$ denotes the completion of Y (a construction that saw a recent surge of interest after work of Dzhafarov (2019)).

Proposition

The summation operator $f \mapsto \Sigma f$ is an interior operator on the Weihrauch lattice.

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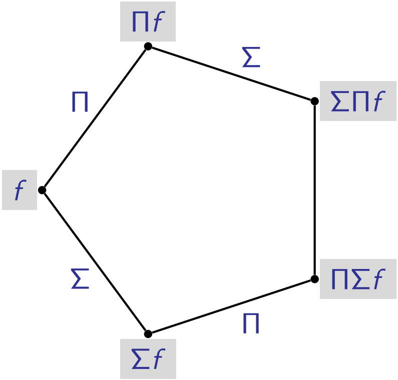
The Parallelization Summation Pentagons

In the general situation parallelization and summation can generate at most five different problems in the Weihrauch lattice:



There are no cross reductions in a proper pentagon (otherwise the pentagon collapses to a smaller graph).

Surprisingly, $\Sigma\Pi f$ and $\Pi\Sigma f$ are always "computability theoretic" problems that can be expressed using Turing cones.

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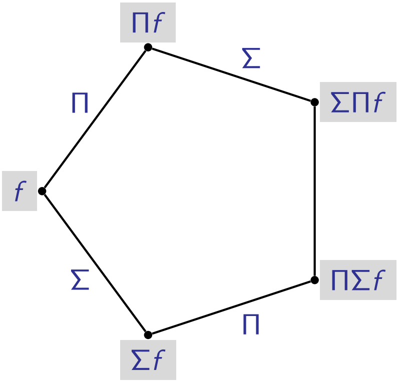

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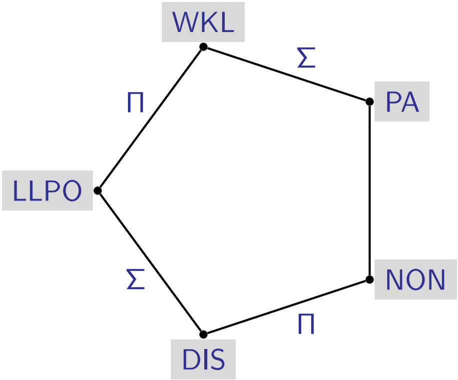
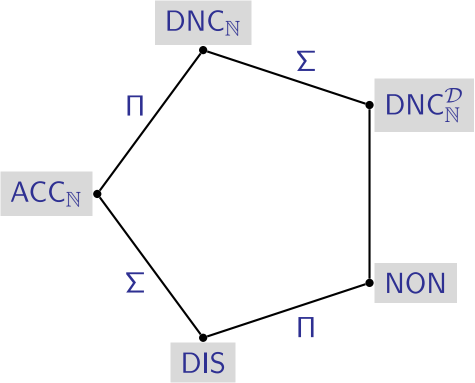


Here ΠLLPO $\equiv_W$ WKL was proved by B. and Gherardi (2011).

WKL denotes Weak Kőnig's Lemma and PA the problem of finding a Turing degree that is of PA degree relative to the given input.

The ACC Pentagon



Here $\Pi\text{ACC}_{\mathbb{N}} \equiv_W \text{DNC}_{\mathbb{N}}$ was proved independently by Higuchi and Kihara (2014) and B., Hendtlass and Kreuzer (2017).

$\text{DNC}_{\mathbb{N}}$ denotes the problem of finding a point in Baire space that is diagonally non-computable relative to the given input.

Conclusion

- We claim that in a well justified way the discontinuity problem DIS can be seen as the weakest natural unsolvable problem.
- The existence of other weak unsolvable problems depends on the axiomatic setting.
- Parallelization of the discontinuity problem DIS yields the non-computability problem.
- Summation of LLPO (and $\text{ACC}_{\mathbb{N}}$ and other problems) yields the discontinuity problem DIS.
- Hence the discontinuity problem is also naturally behaved with respect to the algebraic structure of the Weihrauch lattice.
- All this is work in progress, nothing has been published yet and there are many open questions left.