Weihrauch reducibility around $\Pi^1_1-CA_0$

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In this talk we will consider only the case where $A = \mathbb{N}$ or $A = 2$.

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The following lemma gives a nice relationship between $\mathbb{N}^{\mathbb{N}}$ and $\mathbb{N}^{<\mathbb{N}}$.

Theorem

For any closed set $C \subseteq \mathbb{N}^{\mathbb{N}}$, there exists a tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ such that

$$\forall f (f \in C \iff f \in [T]) \tag{1}$$

Conversely, for any tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$, there exists a closed set $C = [T] \subseteq \mathbb{N}^{\mathbb{N}}$ such that (1) holds. If T is a perfect tree, then C is a perfect set.

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Theorem (Cantor-Bendixson)

Every closed set (tree) in $\mathbb{N}^{\mathbb{N}}$ ($\mathbb{N}^{<\mathbb{N}}$) can be written as $P \cup S$, where P is a perfect closed set (perfect tree) and S is a countable set (countable set of paths).

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A represented space is a a pair (X, δ), where $\delta :\subseteq \mathbb{N}^\mathbb{N} \to X$ is a (possibly partial) surjection. We say that $p \in \mathbb{N}^\mathbb{N}$ is a name for an element of X.

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- Tr: a tree is represented via its characteristic function on the set of finite sequences.
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Definition

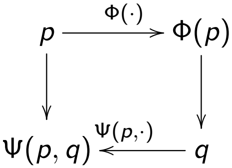
A problem $f :\subseteq X \rightrightarrows Y$ is a (partial) multivalued function between represented spaces.

Weihrauch reducibility

Definition

Let f, g be two problems. f is Weihrauch reducible to g, ($f \leq_{\mathrm{W}} g$), if there are computable $\Phi, \Psi \subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that

- if p is a name for an instance of f, then $\Phi(p)$ is a name for an instance of g;
- if p is a name for an instance of f and q is a name for a solution of g to $\Phi(p)$ then $\Psi(p, q)$ is a name for a solution of f to p



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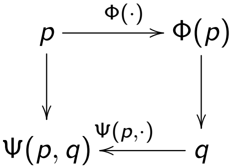

We call Φ the forward function and Ψ the backward one.

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- $\chi_{\Pi^1_1} :\subseteq \mathrm{Tr} \to \{0,1\}$ given in input a tree in $\mathbb{N}^{<\mathbb{N}}$, say whether it is well-founded or not;
- $\Pi^1_1{-}\mathrm{CA} := \widehat{\chi_{\Pi^1_1}}$;
- $\mathrm{PK}_{\mathrm{Tr}} :\subseteq \mathrm{Tr} \to \mathrm{Tr}$ given in input a tree in $\mathbb{N}^{<\mathbb{N}}$, produce its perfect kernel (as a tree);
- $\mathrm{PK}_{\mathbb{N}^\mathbb{N}} :\subseteq \mathcal{A}(\mathbb{N}^\mathbb{N}) \to \mathcal{A}(\mathbb{N}^\mathbb{N})$ given in input a closed set in $\mathbb{N}^\mathbb{N}$, produce its perfect kernel (as a tree whose body is the perfect kernel);
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Some (non) reductions

$\Pi^1_1\text{-CA} \equiv_W \mathsf{PK}_{\mathrm{Tr}}$

Hirst

$\mathsf{PK}_{\mathbb{N}^\mathbb{N}}$

$\chi_{\Pi^1_1}$

$\mathsf{PK}_{2^\mathbb{N}}$

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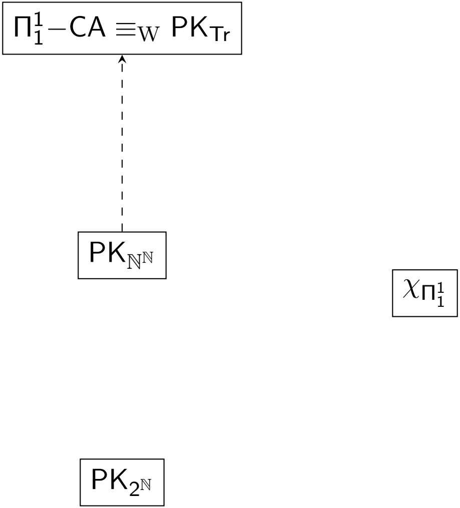

A closed set in $\mathbb{N}^\mathbb{N}$ is represented via a tree T s.t. $C = [T]$.
Then, $\mathsf{PK}_{\mathsf{Tr}}(T)$ is a suitable name for the perfect kernel of a closed set in $\mathbb{N}^\mathbb{N}$.

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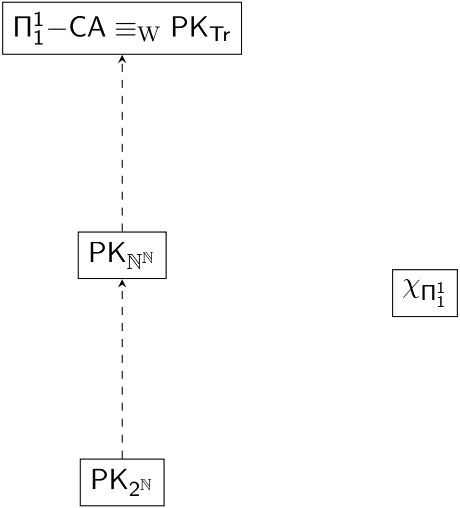

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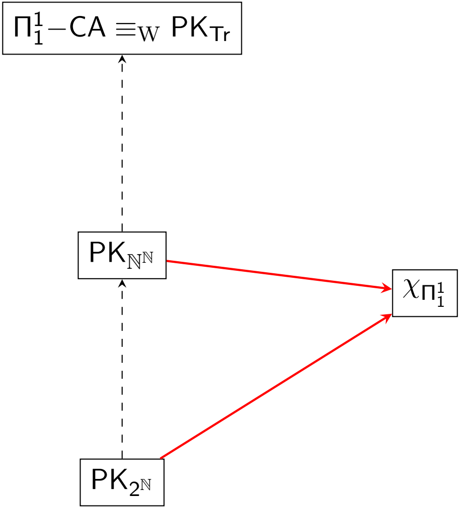

$\chi_{\Pi^1_1}$ has only computable outputs.

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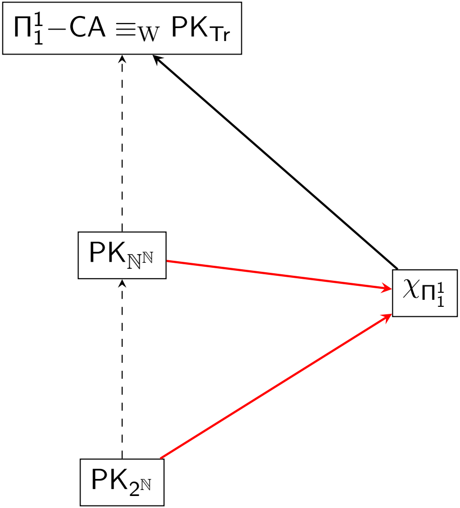

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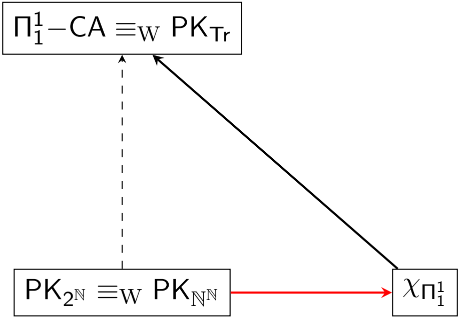

The proof is based on the fact that, given $T \subseteq \mathbb{N}^{<\mathbb{N}}$, we can computably translate T into a tree $T^* \subseteq 2^{<\mathbb{N}}$ via the map $f^* : \mathbb{N}^{<\mathbb{N}} \to 2^{<\mathbb{N}}$ defined as $f^*(\langle x_n \rangle_{n<k}) = 0^{x_0}10^{x_1}1, \dots, 10^{x_{k-1}}$. Then, from $\mathsf{PK}_{2^{\mathbb{N}}}([T^*])$, we can computably obtain the perfect kernel in $\mathbb{N}^{<\mathbb{N}}$.

$\mathsf{PK}_{2^{\mathbb{N}}}$ and $\Pi^1_1-\mathsf{CA}$

LPO: given a sequence $p \in 2^{\mathbb{N}}$, $\mathsf{LPO}(p) = \begin{cases} 1 \text{ if } (\exists k)p(k) = 0 \\ 0 \text{ otherwise} \end{cases}$

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Lemma

Π^1_1-CA *is not closed under compositional product with* LPO.

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Finally we obtain that $\mathsf{PK}_{2^{\mathbb{N}}} <_{\mathrm{W}} \Pi^1_1{-}\mathsf{CA}$.

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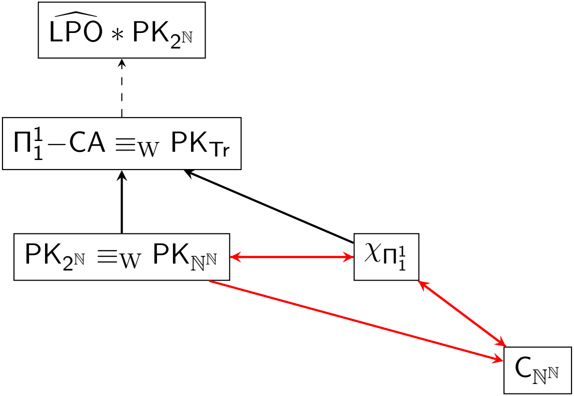
Finally we obtain that $\mathsf{PK}_{2^{\mathbb{N}}} <_{\mathrm{W}} \Pi^1_1-\mathsf{CA}$.
Furthermore, since $\mathsf{PK}_{2^{\mathbb{N}}}$ is parallelizable, we obtain the following:

Lemma

$\chi_{\Pi^1_1} |_{\mathrm{W}} \mathsf{PK}_{2^{\mathbb{N}}}$.



$\mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$: given in input some $C \in \mathcal{A}(\mathbb{N}^{\mathbb{N}})$ output some $p \in C$.
Remark: $\mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$ is closed under compositional product.

How strong is $\mathsf{PK}_{2^{\mathbb{N}}}$?

We first show the reduction with LPO.

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$\langle\rangle$

How strong is $PK_{2^{\mathbb{N}}}$?

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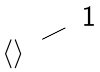

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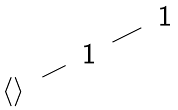

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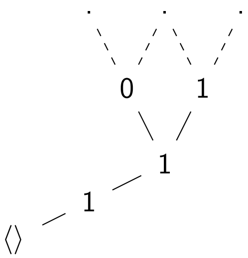

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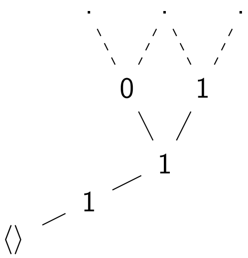

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The following holds:

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Both conditions are Σ^0_1 so we can computably answer the question.

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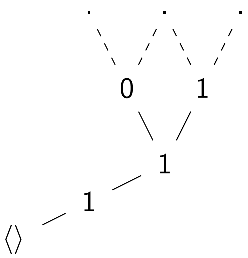

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Both conditions are Σ^0_1 so we can computably answer the question.

Since $\mathrm{PK}_{2^{\mathbb{N}}}$ is parallelizable, $\widehat{\mathrm{LPO}} \leq_{\mathrm{W}} \mathrm{PK}_{2^{\mathbb{N}}}$.

Listing points in the scattered part

We review two principles introduced by Kihara, Marcone and Pauly.

- wList $:\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows (\mathbb{N}^\mathbb{N})^\omega$ maps a countable set A to some $\langle b_0 p_0, b_1 p_1 \dots \rangle$ such that $A = \{p_i : b_i = 1\}$.
- List $:\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows (\mathbb{N}^\mathbb{N})^\omega$ maps a countable set A to some $n\langle p_0, p_1 \dots \rangle$ such that:
  - either $n = 0$ $p_i \neq p_j$ for $i \neq j$ and $A = \{p_i : i \in \mathbb{N}\}$,
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Both Principles are known to be equivalent to $\mathrm{UC}_{\mathbb{N}^\mathbb{N}}$, the principle that, given in input a closed set in $\mathbb{N}^\mathbb{N}$ with a single element, outputs that element.

Cantor-Bendixson Principles

- $\mathsf{CB_{Tr}} :\subseteq \mathsf{Tr} \rightrightarrows \mathsf{Tr} \times (\mathbb{N}^{\mathbb{N}})^{\omega}$.
  Takes as input a tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ and gives as output the perfect kernel of T plus a "strong" list of the scattered part.
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Cantor-Bendixson Principles

- $\mathsf{CB}_{\mathsf{Tr}} :\subseteq \mathsf{Tr} \rightrightarrows \mathsf{Tr} \times (\mathbb{N}^{\mathbb{N}})^{\omega}$.
  Takes as input a tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ and gives as output the perfect kernel of T plus a "strong" list of the scattered part.
- $\mathsf{wCB}_{\mathsf{Tr}} :\subseteq \mathsf{Tr} \rightrightarrows \mathsf{Tr} \times (\mathbb{N}^{\mathbb{N}})^{\omega}$.
- $\mathsf{CB}_{\mathbb{N}^{\mathbb{N}}} :\subseteq \mathcal{A}(\mathbb{N}^{\mathbb{N}}) \rightrightarrows \mathcal{A}(\mathbb{N}^{\mathbb{N}}) \times (\mathbb{N}^{\mathbb{N}})^{\omega}$: takes as input a closed set $C \subseteq \mathbb{N}^{\mathbb{N}}$ represented via a tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ such that $[T] = C$, and gives as output a tree T' such that $[T']$ is the perfect kernel of T plus a "strong" list of the scattered part.
- $\mathsf{wCB}_{\mathbb{N}^{\mathbb{N}}} :\subseteq \mathcal{A}(\mathbb{N}^{\mathbb{N}}) \rightrightarrows \mathcal{A}(\mathbb{N}^{\mathbb{N}}) \times (\mathbb{N}^{\mathbb{N}})^{\omega}$

$\chi_{\Pi^1_1} <_W \mathsf{CB}_{\mathbb{N}^\mathbb{N}}$

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We computably build a new tree $T_0 \subseteq \mathbb{N}^{<\mathbb{N}}$ such that
T is ill-founded $\iff$ T_0 is perfect.

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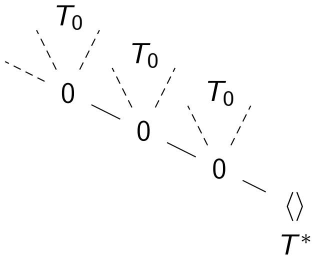

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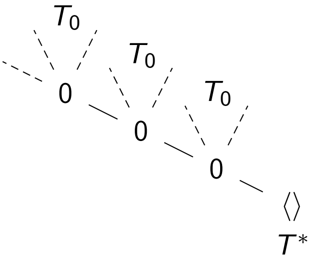

- If T_0 is perfect (T is ill-founded), the scattered part of T^* is empty.
- If T_0 is not perfect (T is well-founded), the perfect kernel of T^* is empty and its scattered part contains a single path (i.e. 0^ω).

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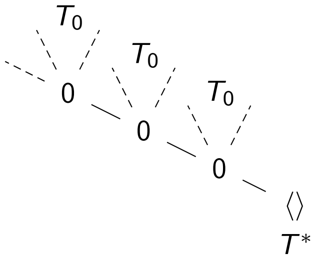

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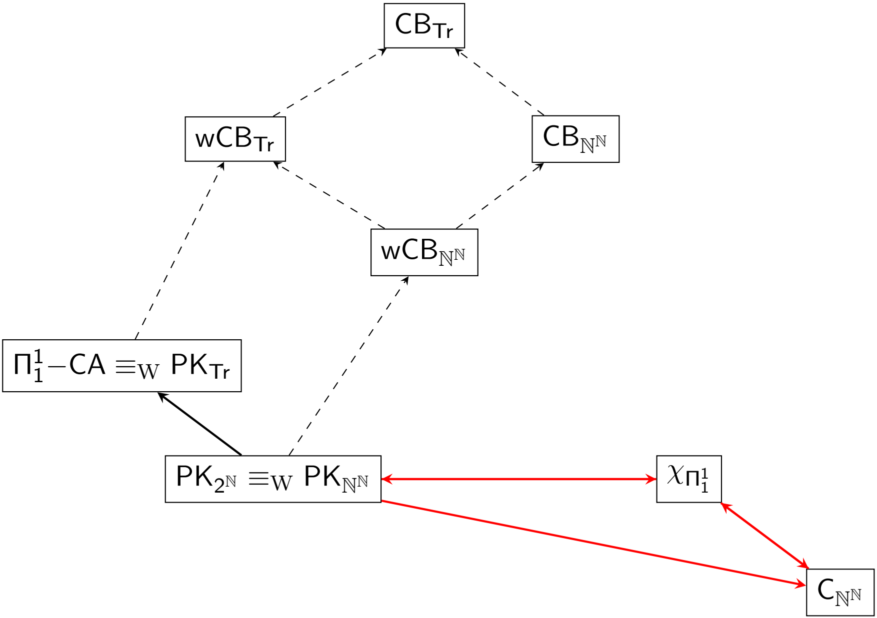
Since the list of the scattered part is "strong", check the number n of paths.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$\Pi^1_1\text{-}\mathsf{CA} \equiv_W \mathsf{PK}_{\mathsf{Tr}}$
$\mathsf{PK}_{2^\mathbb{N}} \equiv_W \mathsf{PK}_{\mathbb{N}^\mathbb{N}}$
$\chi_{\Pi^1_1}$
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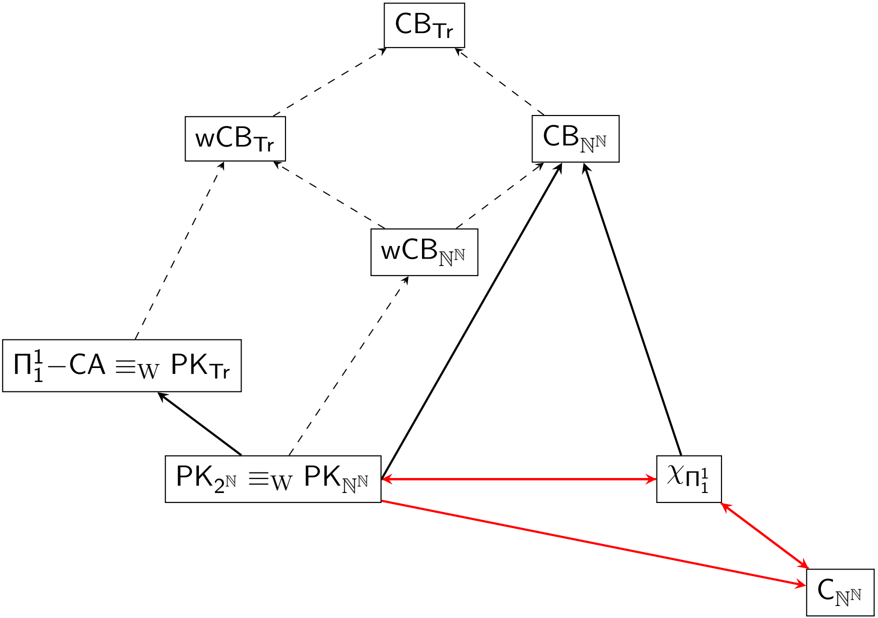

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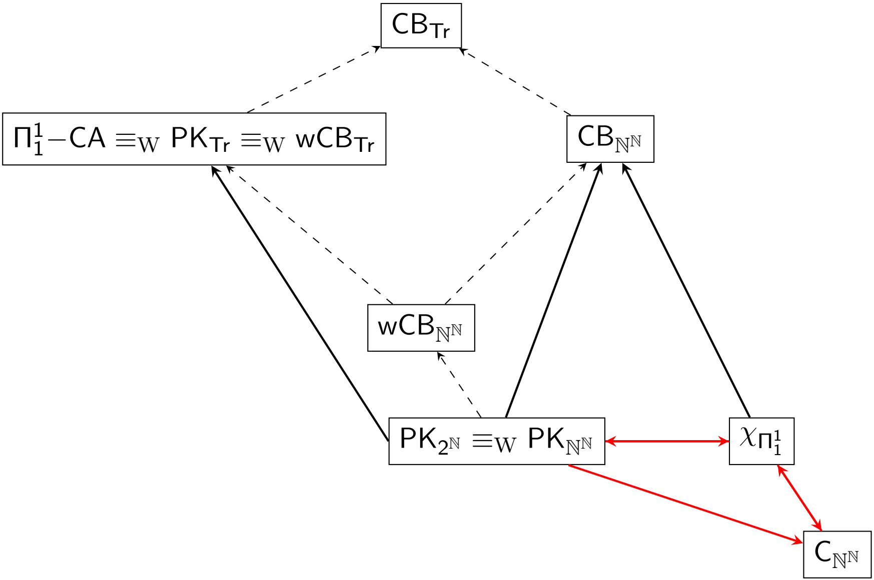

Following a theorem by Kreisel, we obtain $\mathsf{wCB}_{\mathsf{Tr}} \equiv_W \Pi^1_1\text{–CA}$

Listing the scattered part in Tr

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- $\mathsf{wList} :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows (\mathbb{N}^{\mathbb{N}})^{\omega}$ maps a countable set A to some $\langle b_0 p_0, b_1 p_1 \dots \rangle$ such that $A = \{p_i : b_i = 1\}$.
- $\mathsf{List} :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows (\mathbb{N}^{\mathbb{N}})^{\omega}$ maps a countable set A to some $n\langle p_0, p_1 \dots \rangle$ such that:
  - either $n = 0$ $p_i \neq p_j$ for $i \neq j$ and $A = \{p_i : i \in \mathbb{N}\}$,
  - or $n > 0$, $|A| = n - 1$ and $A = \{p_i : i < n - 1\}$.

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- $\mathsf{prwList} :\subseteq \mathsf{Tr} \rightrightarrows (\mathbb{N}^{\mathbb{N}})^{\omega}$ maps a pruned tree T with countably many paths to some $\langle b_0 p_0, b_1 p_1 \ldots \rangle$ such that $[T] = \{p_i : b_i = 1\}$.
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prwList and prList

Recently, Dzhafarov, Solomon, Yokoyama defined the first-order part of a problem f (denoted by ${}^1f$), is defined as $sup_{\leq_W}\{f_0 :\subseteq Z \to \mathbb{N} : f_0 \leq_W f\}$.

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Lemma

$\mathsf{CB}_{\mathsf{Tr}} \leq_W$ prList $*$ $\mathsf{PK}_{\mathsf{Tr}}$.

$\mathsf{CB}_{\mathsf{Tr}} \le_{\mathrm{W}} \mathsf{prList} * \mathsf{PK}_{\mathsf{Tr}}$

Let T be an input for $\mathsf{CB}_{\mathsf{Tr}}$. We (computably) build two sequence of trees

- $(T_\sigma)_{\sigma \in T}$;
- $(T^\sigma)_{\sigma \in T}$;

where;

- $[T_\sigma] \neq \emptyset \iff \sigma$ is contained in a perfect subtree of T;
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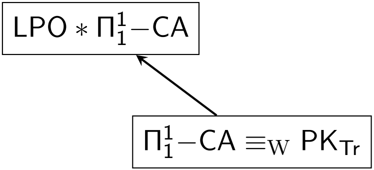
From $S = \{\sigma \in T : \sigma \in Z'\}$ we can derive the scattered part of T.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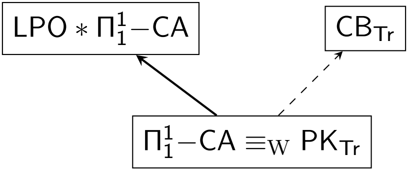
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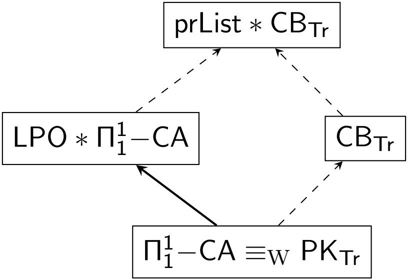
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Thanks for Your attention!