AN EFFECTIVE COMPLETENESS THEOREM FOR COMPUTABLE PRESENTATIONS AND CONTINUOUS LOGIC

CALEB CAMRUD* AND TIMOTHY H. MCNICHOLL

Within model theory, a natural question to ask is "Which model-theoretic results have effective counterparts?" A wide range of results have been proven in this area with respect to classical logic, surveyed in [1]. Classical model theory, however, has limitations in its application to non-discrete metric structures. Circumventing these limitations, Ben Yaacov et al. developed a model theory for metric structures using continuous first-order logic [2]. A completeness result for this logic was proven [3] soon after. Following this, Calvert proved a version of an effective completeness theorem for continuous logic utilizing probabilistic computation [4]. Resurrecting this decade-old project in a new light, we examine the effective model-theoretic properties of continuous first-order logic within the framework of computable presentations. Our main result is a new version of the effective completeness theorem: any decidable continuous first-order theory has a computably presentable model.

The well-formed formulas and sentences of continuous logic mirror those found in classical logic, with three key differences. First, the metric d subsumes the role of $=$, with $p = q$ if and only if $d(p, q) = 0$. Because of this, a structure $\mathfrak{M}$ satisfies a sentence φ if and only if $\varphi^{\mathfrak{M}} = 0$. Second, instead of the classical logical connectives $\neg$ and $\wedge$, the set of logical connectives includes symbols to represent all continuous functions, of any finite arity, mapping $[0, 1]$ into itself. However, to examine computability on this logic, a countable subset of well-formed formulas is considered, instead. As such, the standard convention considers only the logical connectives $\left\{\neg, \dot{-}, \frac{1}{2}\right\}$, which are interpreted as $(\neg\varphi)^{\mathfrak{M}} = 1 - \varphi^{\mathfrak{M}}$, $(\varphi \dot{-} \psi)^{\mathfrak{M}} = \max\left\{0, \varphi^{\mathfrak{M}} - \psi^{\mathfrak{M}}\right\}$, and $\left(\frac{1}{2}\varphi\right)^{\mathfrak{M}} = \frac{1}{2} \cdot \varphi^{\mathfrak{M}}$. It was shown in [5] that this set of connectives generates a dense set in the space of connectives of each arity. Lastly, instead of the classical quantifiers $\forall$ and $\exists$, the quantifiers of continuous logic are sup and inf. Since satisfaction occurs only if $\varphi^{\mathfrak{M}} = 0$, sup acts as a "universal" quantifier, and inf as an "existential" quantifier (though there need not always be a witness for satisfaction to occur, only a sequence of approximate witnesses).

The axioms of continuous logic are provided in [3]. A consistent set of sentences is called a *theory*. Given a theory T, the *degree of truth with respect to T* is a function $\cdot\,^{\circ}_{T}$ defined on the set of all sentences as

$$\varphi^{\circ}_{T} := \sup\left\{\varphi^{\mathfrak{M}} : \mathfrak{M} \vDash T\right\}.$$

A theory T is then *decidable* if $\cdot\,^{\circ}_{T}$ is a computable function from the set of sentences to $[0, 1]$.

Our definition of a metric structure and signature follow that given in [2] and later [6], though we retain the classical condition from [2] that the metric space within the structure is bounded. Given a metric structure $\mathfrak{M}$ over a metric space (M, d) and $A \subseteq M$, the *algebra generated by A* is the smallest subset of M which contains A and is closed under every function of $\mathfrak{M}$. A *presentation* of $\mathfrak{M}$ is a pair $\big(\mathfrak{M}, \{p_n\}_{n\in\mathbb{N}}\big)$ such that $\{p_n : n \in \mathbb{N}\} \subseteq M$ and the algebra generated by $\{p_n : n \in \mathbb{N}\}$ is dense in M. Given a presentation $\mathfrak{M}^\sharp = \big(\mathfrak{M}, \{p_n\}_{n\in\mathbb{N}}\big)$, we call p_n the *n-th distinguished point* of $\mathfrak{M}^\sharp$, and the algebra generated by $\{p_n : n \in \mathbb{N}\}$ the set of *rational points* of $\mathfrak{M}^\sharp$, denoted $\mathbb{Q}(\mathfrak{M}^\sharp)$.

A presentation $\mathfrak{M}^\sharp = \big(\mathfrak{M}, \{p_n\}_{n\in\mathbb{N}}\big)$ is *computable* if $\mathfrak{M}$ has a computable signature and the metric and every predicate of $\mathfrak{M}$ is computable, uniformly on the rational points of $\mathfrak{M}^\sharp$. That is, $\mathfrak{M}^\sharp$ is computable if both of the following conditions hold.

(1) There is an algorithm such that, given any $q_0, q_1 \in \mathbb{Q}(\mathfrak{M}^\sharp)$ and $k \in \mathbb{N}$, computes a rational number r such that $\big|r - d(q_0, q_1)\big| < 2^{-k}$.

(2) For every n-ary predicate R of $\mathfrak{M}$, there is an algorithm such that, given any $q_0, ..., q_{n-1} \in \mathbb{Q}(\mathfrak{M}^\sharp)$ and $k \in \mathbb{N}$, computes a rational number r such that $\big|r - R(q_0, ..., q_{n-1})\big| < 2^{-k}$.

The following is our primary result.

Theorem 1. *For any decidable theory T, there is a signature L and a computably presentable metric L-structure $\mathfrak{M}$ such that $\mathfrak{M} \vDash T$.*

The sketch of the proof is as follows, beginning similarly to that found in [4]. Fix a decidable theory T. At stage s, use the computability of $\cdot \overset{\circ}{_T}$ to construct a finite, relatively consistent $\big($up to a value of $2^{-(s+1)}\big)$, pre-Henkin complete set which includes part of T. When this procedure is correctly defined, in the limit it produces a maximally consistent, Henkin complete extension of T. By the standard Henkin model construction, define $\mathfrak{M}$ from this set. Then $\mathfrak{M}$ is an L-structure which satisfies T. Define a presentation which includes as designated points the interpretations of the added Henkin constants. Then, given a tuple of rational points of $\mathfrak{M}$, query a large enough one of the pre-Henkin complete sets previously computed to find a rational number arbitrarily close to the value of the metric or any other predicate from $\mathfrak{M}$ on those points. It follows that the presentation is computable.

References

[1] V. S. Harizanov, *Pure computable model theory*, ***Handbook of Recursive Mathematics, vol. 1***, (Yu. L. Ershov, S. S. Goncharov, A. Nerode, J.B. Remmel, and V. W. Marek, editors) Studies in Logic and the Foundations of Mathematics, 138-139, North-Holland, Amsterdam, 1998, pp. 3–114.

[2] Itaï Ben Yaacov, Alexander Berenstein, C. Ward Henson, and Alexander Usvyatsov, *Model theory for metric structures*, ***Model Theory with Applications to Algebra and Analysis, vol. 2***, (Zoé Chatzidakis, Dugald Macpherson, Anand Pillay, and Alex Wilkie, editors) London Math Society Lecture Note Series, vol. 350, London, 2008, pp. 315–427.

[3] Itaï Ben Yaacov and Arthur Paul Pedersen, *A proof of completeness for continuous first-order logic*, ***The Journal of Symbolic Logic***, vol. 75 (2010), pp. 168–190.

[4] Wesley Calvert, *Metric structures and probabilistic computation*, ***Theoretical Computer Science***, vol. 412 (2011), pp. 2766–2775.

[5] Itaï Ben Yaacov and Alexander Usvyatsov, *Continuous first order logic and local stability*, ***Transactions of the American Mathematical Society***, vol. 362, no. 10 (2010), pp. 5213–5259.

[6] Johanna N.Y. Franklin and Timothy H. McNicholl, *Degrees of and lowness for isometric isomorphism*, ***Journal of Logic & Analysis***, vol. 12, no. 6 (2020), pp. 1–23.

Department of Mathematics, Iowa State University, Ames, Iowa 50011
Email address: ccamrud@iastate.edu