Cantor-Bendixson Theorem and Weihrauch reducibility

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In this talk, using the framework of Weihrauch reducibility, we aim to study theorems that are at the very high levels of reverse mathematics strength. In [4], the authors considered multi-valued functions (or problems) related to statements that in reverse mathematics are known to be equivalent to ATR_0. Recently, there has been a growing interest in this topic and several authors explored other problems around ATR_0, showing that "natural" candidates of ATR_0 may have different uniform computational strength(see for example [5], [2], [7], [8]). See [6] and [9] for a summary on these results.
Starting from one of the problems considered in [4], namely PTT_1 (the problem of finding a perfect subtree of a tree with uncountable body), we show that if instead of trees we consider closed sets, we obtain a strictly weaker problem. This difference disappears if we use arithmetical Weihrauch reducibility instead of the classical one. The main part of the talk will focus on (variations of) the Cantor-Bendixson theorem, that in reverse mathematics are equivalent to Π^1_1-CA_0. The natural problem that represents Π^1_1-CA_0, is the one that maps a countable sequence of trees to the characteristic function of the set of indices corresponding to well-founded trees and that we denote with Π^1_1. Recently in [3], Hirst showed its Weihrauch equivalence with $\mathsf{PK}_{\mathsf{Tr}}$, the function that takes as input a tree and outputs its perfect kernel. As for perfect sets, here the variant for trees is strictly stronger than the one for closed sets (denoted PK_X, with X a computable Polish space), but arithmetically Weihrauch equivalent to it. Finally we consider problems that in addition to the perfect kernel, also require a listing of the points in the scattered part. We will show that, for trees, these principles are equivalent to Π^1_1, while if we consider closed sets (either in Cantor or Baire space) they are at the same level of PK_X, with the only exception of $\mathsf{CB}_{\mathbb{N}^\mathbb{N}}$ that is strictly above PK_X and strictly below Π^1_1.
This is joint work with Alberto Marcone and Manlio Valenti.

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