AN ALGORITHM FOR REORIENTING INFINITE PSEUDO-TRANSITIVE GRAPHS

MARTA FIORI CARONES
(JOINT WITH ALBERTO MARCONE)

In 1962 Alain Ghouila-Houri proved the following theorem in [GH62]:

Theorem 1. *Every pseudo-transitive oriented graph has a transitive reorientation.*

Oriented graphs are directed graphs such that at most one of the edges between two vertices exist. An oriented graph $(V, \to)$ is *pseudo-transitive* if for every $a, b, c \in V$ such that $a \to b$ and $b \to c$ we have also either $a \to c$ or $c \to a$. An oriented graph $(V, \to)$ is *transitive* if for every $a, b, c \in V$ such that $a \to b$ and $b \to c$ we have also $a \to c$, that is $(V, \to)$ is a partial order. A *reorientation* (V, R) of an oriented graph $(V, \to)$ is an oriented graph obtained by reversing some of the edges, so that $a, b \in V$ are $\to$-comparable if and only if they are R-comparable.

Ghouila-Houri's proof of Theorem 1 deals only with finite graphs and uses induction on the number of vertices. The same proof is presented in [Ber76, Fis85, Har05] and extended to the infinite case by some compactness argument.

Ghouila-Houri proved the previous theorem as the main step towards the proof of a characterisation theorem for comparability graphs, i.e. graphs which admit a transitive orientation, where $\to$ is an *orientation* of (V, E) if it is an asymmetric and irreflexive relation and if for every $a, b \in V$ we have $a\,E\,b$ if and only if $a \to b$ or $b \to a$. In fact, he proved the following:

Theorem 2. *An undirected graph has a transitive orientation if and only if every cycle of odd length has a triangular chord.*

While the effectiveness of the latter theorem was already studied, in particular using the framework of reverse mathematics, by Jeff Hirst in his PhD thesis [Hir87], we concentrate on the effectiveness of the former theorem, obtaining a result which can be stated in various different ways.

Theorem 3.

1. RCA_0 *proves that every countable pseudo-transitive oriented graph has a transitive reorientation;*
2. *every computable pseudo-transitive oriented graph has a computable transitive reorientation;*
3. *the multi-valued function that maps a countable pseudo-transitive oriented graph to the set of its transitive reorientations is computable;*
4. *there exists an on-line (incremental) algorithm to transitively reorient pseudo-transitive oriented graphs;*
5. *Player II has a winning strategy for the following game: starting from the empty graph, at step $s+1$ player I plays a pseudo-transitive extension $(V_s \cup \{x_s\}, \to_{s+1})$ of the pseudo-transitive oriented graph $(V_s, \to_s)$ he played at step s. Player II replies with a transitive reorientation $\prec_{s+1}$ of $\to_{s+1}$*

such that $\prec_{s+1}$ *extends* $\prec_s$ *she defined at step* s*. Player II wins if and only if she is always able to play.*

Our algorithm applies to oriented graphs of any cardinality, as long as the set of vertices can be well-ordered.

Restricted to countable graphs, our result shows in particular that compactness is not necessary to prove Theorem 1, while Hirst showed that it is necessary to prove Theorem 2.

References

[Ber76] Claude Berge. *Graphs and Hypergraphs*. North-Holland, Amsterdam, 2nd edition, 1976.

[Fis85] Peter C. Fishburn. *Interval Orders and Interval Graphs*. Wiley, New York, 1985.

[GH62] Alain Ghouila-Houri. Caractérisation des graphes non orientés dont on peut orienter les arĕtes de manière à obtenir le graphe d'une relation d'ordre. *Les Comptes Rendus de l'Académie des Sciences*, 254:1370–1371, 1962.

[Har05] Egbert Harzheim. *Ordered Sets*. Springer, New York, 2005.

[Hir87] Jeffry L. Hirst. *Combinatorics in Subsystems of Second Order Arithmetic*. PhD thesis, The Pennsylvania State University, 1987.