Coloring Subsets of Euclidean Space

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It is well known that no proper nonempty subsets of $\mathbb{R}^n$ can give a clear answer to the query which asks if the point is in the set or not, within a finite time. [1, Theorem 5.1.5] Therefore, it is known that for some points, we can't verify whether the point is in or out of the set. In other words, in order to finish answering such queries of points in $\mathbb{R}^n$ in finite time, some points should be answered as unknown. Because of this uncertainty, we should accept two facts to draw a set.

- Two different sets may be drawn the same.
- One set may be drawn in several different ways.

And they can be formalized by the following definitions. Consider that drawing is a process of filling each pixels with some colors.

Definition 1 (Resolution, d-xel)**.** Resolution $r = (r_1, \cdots, r_d)$ is a tuple of d positive integers, which denotes the grid in $[0, 1]^d$ of width $1/r_i$ for each dimension. Denote the set of resolutions as $\mathcal{R}_d$.

d-xel is a d-dimensional version of pixel or voxel. Each d-xel is represented as a cartesian product of intervals. Denote the set of d-xels as $\mathcal{P}_r$.

$$\mathcal{P}_r := \left\{ \prod_{i=1}^{d} \left[\frac{a_i}{r_i}, \frac{a_i + 1}{r_i} \right] \middle| a_i \in ([0, r_i - 1] \cap \mathbb{N}) \right\}$$

The (d-)volume of a d-xel is $1/\prod_{i=1}^{d} r_i$.

Definition 2 (Color function)**.** For $S \subseteq \mathbb{R}^d$, $C_S : \{(r, P)|r \in \mathcal{R}_d, P \in \mathcal{P}_r\} \to \{-1, 0, 1\}$ is called a color function of S if it satisfies the following.

For any resolution $r \in \mathcal{R}_d$ and a d-xel $P \in \mathcal{P}_r$,

- If $C_S(r, P) = 1$, then $P \cap S \neq \emptyset$
- If $C_S(r, P) = -1$, then $P \cap S^c \neq \emptyset$
- If $C_S(r, P) = 0$, then there exists $a \in P$ s.t. $d(a, \partial S) < 1/\min_i r_i$ where d is the Hausdorff distance.

With this definition, every subset of $\mathbb{R}^n$ has at least one color function. However, it doesn't mean that every set can be colored. Consider $\mathbb{Q} \times \mathbb{Q} \subseteq \mathbb{R}^2$. With any resolution, every pixel can be colored 1, 0, or -1 because $\partial(\mathbb{Q} \times \mathbb{Q}) = \mathbb{R}^2$. Intuitively, we don't want to say that randomly colored pixel is a drawing of a set. To avoid this, we need additional constraint.

Definition 3. The set S is colorable if, for every color function C_S of S, the sum of volume of 0-colored d-xels goes to zero as $\min_i r_i$ goes to infinity.

The colorable property has a significant connection with the measure theory.

Theorem 1. *The set S is colorable iff $\lambda(\partial S) = 0$ where λ is standard Lebesgue measure of $\mathbb{R}^d$.*

There are also some notable properties of colorable sets.

Proposition 1. *The set is colorable if and only if its complement is colorable.*

With this classification, it is clear that finite union of colorable sets are colorable. However, for the infinite case, there is a counterexample.

Example 1. $\{r/2^n | r, n \in \mathbb{N}\}$*, is not colorable even if it is a countable union of colorable sets.*

And surprisingly, there exists a non-colorable curve in $\mathbb{R}^2$, because there exists a curve with positive measure.

Example 2. *The image of Osgood curve [2] in $\mathbb{R}^2$ is not colorable.*

Finally, we define a computably colorable set, since the definition of colorable set does not include any computational property.

Definition 4 (Computably Colorable set)**.** The colorable set S is computably colorable in $[0, 1]^d$ if there is a color function C_S of S which is computable.

Example 3. *For non-computable number $r \in (0, 1)$, $[0, r]^d$ is colorable but is not computably colorable, for any d.*

We will also explain how this concept is related to the representation of subsets of Euclidean space, and give out algorithms to draw such sets, which is the generalization of the result in [3].

References

[1] Klaus Weihrauch. *Computable analysis: an introduction.* Springer Science & Business Media, 2012.

[2] William F Osgood. A jordan curve of positive area. *Transactions of the American Mathematical Society*, 4(1):107–112, 1903.

[3] Jiman Hwang and Sewon Park. Compact subsets in exact real computation. *한국정보과학회 학술발표논문집*, pages 1104–1106, 2020.