On Representations of Irrational Numbers: A Degree Structure

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There are numerous ways to represent real numbers. We may use, e.g., nested intervals, Cauchy sequences (with modulus of convergence), Dedekind cuts, numerical base-10 expansions, numerical base-2 expansions and continued fractions. All these representations are equivalent in the sense that we can efficiently convert one representation into another representation, more precisely:

> *Let R_1 and R_2 be any of the representations mentioned above. We can compute the R_1-representation of the irrational number α using the R_2-representation of α as an oracle.*

Above we use the word "compute" in the standard way. We are talking about full Turing computability and that includes access to unbounded search. If we do not have access to full Turing computability and unbounded search, then we cannot in general convert one representation into another. Our investigations are motivated by the following question:

> *Do we need, or do we not need, unbounded search in order to convert one representation of an irrational number into another representation?*

A computation that does not apply unbounded search is called a *subrecursive computation*. Primitive recursive computations and (Kalmar) elementary computations are typical examples of subrecursive computations. So are polynomial-time computations.

Definition 1. *Let R_1 and R_2 be representations of irrational numbers. The relation $R_1 \leq_{\mathcal{S}} R_2$ (R_1 is $\mathcal{S}$-reducible to R_2) holds iff the R_1-representation of α can be subrecursively computed in the R_2-representation of α. The relation $R_1 \equiv_{\mathcal{S}} R_2$ (R_1 is $\mathcal{S}$-equivalent to R_2) holds iff $R_1 \leq_{\mathcal{S}} R_2$ and $R_2 \leq_{\mathcal{S}} R_2$. The relation $R_1 <_{\mathcal{S}} R_2$ holds iff $R_1 \leq_{\mathcal{S}} R_2$ and $R_2 \not\leq_{\mathcal{S}} R_2$.*

Definition 1 should be considered as a quasi-formal definition since we have not formally defined what a representation is, neither have we formally defined what a subrecursive computation is. The author has a pretty good idea of what a full formal definition $\mathcal{S}$-reducibility should look like, but such a definition will be lengthy and technical.

The following strictly increasing chain is well known:

$$\begin{aligned} \text{Weihrauch intervals} \quad <_{\mathcal{S}} \quad \text{Cauchy sequences} \quad <_{\mathcal{S}} \quad \text{base-10 expansions} \\ <_{\mathcal{S}} \quad \text{Dedkind cuts} \quad <_{\mathcal{S}} \quad \text{continued fractions.} \end{aligned}$$

The equivalence relation $\equiv_{\mathcal{S}}$ induces a degree structure on the representations of irrational numbers. In my talk, which is vaguely based on the publications in the reference list, I will discuss this degree structure. To the extent time permits, I will also discuss a number of less well-know representations as, e.g., base-b sum approximations and general sum approximations (from above and below)[2, 3], left and right best approximations [3], standard and dual Baire sequences [5], trace functions and contractors [1, 5], Hurwitz characteristics and fuzzy Dedekind cuts [4].

References

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