When is the Scott topology countably based?

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Given a topological space X we investigate when the space of open sets $\mathcal{O}(X)$ is countably based, where $\mathcal{O}(X)$ is endowed with the Scott topology induced by $(\mathcal{O}(X), \subset)$ [Bre13]. The countably based property is indeed a key property, since it is a characterisation of the existence of an admissible open representation, which enables to develop descriptive complexity theory over topological spaces. More specifically, we work with X admissibly represented, or equivalently, a qcb $T0$-space.

Some characterisations are already established in particular cases. In case the space X is Hausdorff, $\mathcal{O}(X)$ being countably based is equivalent to the existence of a countable pseudobase that consists of compact sets of X [Sch04]. In case the space X is countably based, $\mathcal{O}(X)$ being countably based is equivalent to the core-compactness of X [Hoy21]. So core-compactness is a key property for understanding when $\mathcal{O}(X)$ is countably based and we want to better understand it.

We prove that in general the existence of a countable pseudobase of compact sets of X implies that $\mathcal{O}(X)$ is countably based but not the converse, providing a counter-example. In particular, X being countably based implies $\mathcal{O}(\mathcal{O}(X))$ (endowed with the Scott topology) being countably based.

When X is the inductive limit of admissibly represented spaces $(X_n)_{n\in\mathbb{N}}$ with each $\mathcal{O}(X_n)$ being countably based, we prove that $\mathcal{O}(X)$ is countably based. We prove this by embedding the space $\mathcal{O}(X)$ in the product space $\prod_{n\in\mathbb{N}} \mathcal{O}(X_n)$. Along the way, we show that the canonical representation of the inductive limit of upward closed admissibly represented spaces is admissible. It is a weakening of the $T1$ hypothesis of [Sch02] Theorem 19.

We investigate the relationship between core-compactness and local compactness. It is already well known that for sober spaces, they are equivalent. We prove that for sub-spaces of $[0\,;\,1]^2$ equipped with the product topology of the Scott topology induced by $([0\,;\,1], <)$, core-compactness is equivalent to local compactness. In the near future, we aim to generalise this result from $[0\,;\,1]^2$ to $[0\,;\,1]^n$. A non-Borel example of core-compact, non-locally compact space has already been found [Gou19; HL78]. We observe that it can be improved to a Borel set of complexity Δ^0_4. A further question is the minimal complexity of a core-compact, non-locally compact space.

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