Predicative Bishop-Cheng Measure Theory

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Abstract

There are two, quite different approaches to classical measure theory within Bishop-style constructive mathematics. The first, introduced in [1], is the so-called *Bishop Measure Theory* (BMT), developed as an abstraction of the measure and integration theory of locally compact metric spaces. Following the standard treatment of integration in classical measure theory, measure is treated in BMT as a primitive notion, and integration is defined with respect to a given measure. The introduction of *complemented subsets* by Bishop was instrumental to overcome in BMT the difficulties generated in measure theory by the use of negation and negatively defined concepts, and it is one of Bishop's great conceptual achievements. At the same time, his extensive use of the concept of a *set-indexed family* of complemented subsets was crucial to the predicative character of BMT. As the set of Borel sets generated by a given family of complemented subsets of a set X, with respect to a set Φ of real-valued functions on X, is inductively defined in [1], p. 68, BMT is based on BISH*, which is Bishop's informal system of constructive mathematics BISH extended with inductive definitions with rules of countably many premises.

The second approach, introduced in [2] and seriously extended in [3], is the so-called *Bishop-Cheng Measure Theory* (BCMT) that makes no use of Borel sets and of no other inductive definitions, and therefore is based only on BISH. Following the tradition of Daniell integration [6], which was taken further by Weil [17], Kolmogoroff [8], Carathéodory [4], and Segal [13], [14], in BCMT Bishop and Cheng consider the integral on a certain set of functions as the primitive notion, extend it to an appropriate, larger set of functions, and then define the measure at a later stage. Although complemented subsets are first-class citizens of BCMT too, their set-indexed families are not. Instead, fundamental concepts of BCMT, like that of integration space and of the set of integrable functions L^1, are defined in an impredicative way, a fact that makes the extraction of the computational content of BCMT and the implementation of BCMT in some programming language impossible. This is why later approaches to measure theory within BISH, like these of Coquand and Palmgren [5], Spitters [16] and Ishihara [7], are developed in an abstract setting that involves certain boolean rings or vector lattices.

In this talk we give an overview of a predicative redevelopment of BCMT within BST, a reconstruction of Bishop's set theory, elaborated in [9], [10]. First, we present the basic theory of complemented subsets, explaining also Shulman's comment in [15] on their relation to the categorical Chu construction (this is expanded in [11]). Next we introduce pre-integration spaces, a predicative reformulation of integration spaces that uses set-indexed families of real-valued partial functions. The pre-integration space of a locally compact metric space is sketched. Based on joint work with Max Zeuner [12] and on his work [18], we describe the pre-integration space of simple functions and of canonically integrable functions, providing a predicative solution to the L^1-completion of an integration space. Finally, we introduce pre-measure spaces, a predicative reformulation of measure spaces in BCMT, and connected them to pre-integration spaces.

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