Quantitative translations for viscosity approximation methods

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Abstract. Proof mining is a research program that employs proof theoretical tools to obtain additional information from mathematical results. Its techniques have been applied successfully to many areas of Mathematics with special focus on Nonlinear Analysis. One well-known strongly convergent algorithm in Fixed Point Theory is due to Halpern [1]. Let X be a Banach space and C a nonempty, closed and convex subset. Consider T a nonexpansive map on C, and $(\alpha_n) \subset [0,1]$ a sequence of real numbers. With $u \in C$ (the anchor point), and $x_0 \in C$, the Halpern iteration is defined by

$$x_{n+1} := \alpha_n u + (1-\alpha_n)T(x_n). \tag{H}$$

The strong convergence of (H) towards a fixed point of T has been extensively studied and was generalized in several ways. Introduced by Moudafi [2], the viscosity approximation method is one such generalization in which the anchor point is replaced by a strict contraction map ϕ,

$$x_{n+1} := \alpha_n \phi(x_n) + (1-\alpha_n)T(x_n). \tag{vH}$$

The study of these iterations is highly relevant with applications in many practical optimization problems. In [3], Suzuki showed that the convergence of the generalized viscosity version (vH) can be reduced to the convergence of the original iteration (H). In such results, one usually looks for metastability rates, i.e. a function φ satisfying

$$\forall \varepsilon > 0 \forall f \in \mathbb{N}^{\mathbb{N}} \exists n \leq \varphi(\varepsilon, f) \forall i,j \in [n, f(n)] \left(\|x_i - x_j\| \leq \varepsilon \right), \tag{1}$$

or for Cauchy rates (when the rate does not depend on the function f). The quantitative analysis of Suzuki's result concerns a $\Pi^0_3 \to \Pi^0_3$ statement and its proof-theoretical interpretation translates classically (via a negative translation) into a transformation of a rate of metastability for the original iteration into a metastability rate for the viscosity version. Such quantitative transformation is explicitly extracted and, furthermore, the analysis reveals that one of Suzuki's conditions was superfluous. Moreover, the quantitative result largely also holds in a geodesic setting. These qualitative improvements point to the success of Proof mining in the generalization of proofs. By an inspection of the quantitative proof, it is possible to see that if we begin with a Cauchy rate for the Halpern iterations, the quantitative transformation will output also a Cauchy rate for the viscosity versions. This entails that Suzuki's arguments are essentially constructive which was not *a priori* obvious. This is in line with previous observations

by Kohlenbach (e.g. in [4]) that by interpreting a proof (even a constructive one) in a classical way one obtains stronger results without any loss of information. We give some examples of application of the quantitative result.

Lastly, we discuss a particular instance where it is possible to obtain a Cauchy rate for the Halpern iteration (H). In many cases, it is possible to obtain rates of convergence for $\|x_n - T(x_n)\| \to 0$, usually called rates of asymptotic regularity. It is well-known that in those situations, a modulus of uniqueness entails the existence of a Cauchy rate for (x_n). In [5], Gwinner introduced the following notion of a uniform accretive operator (here $A = \mathrm{Id} - T$, for T a nonexpansive map on C): there is a strictly increasing function $\varphi : [0,\infty) \to \mathbb{R}$ such that $\lim_{t\to\infty} \varphi(t) = \infty$ and

$$\forall x, y \in C\, \exists j \in J(x-y)\, (\langle A(x) - A(y), j\rangle \geq (\varphi(\|x\|) - \varphi(\|y\|)) \cdot (\|x\| - \|y\|)\,), \quad (2)$$

where J is the normalized duality map of X. Under this assumption, in the setting of uniformly convex Banach spaces, we extract of a modulus of uniqueness from Gwinner's uniqueness proof. Using a previous rate of asymptotic regularity, we finish the talk with an application of this result to obtain a Cauchy rate for the Halpern iteration (and thus also for the viscosity version) when $\alpha_n = \frac{1}{n+1}$.

Keywords. Viscosity method, Proof mining, Rates of convergence, Metastability.

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