COMPUTABILITY OF WEAK CONVERGENCE OF MEASURES ON THE REAL LINE

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Given a separable metric space X, a sequence $\{\mu_n\}_{n\in\mathbb{N}}$ of finite Borel measures on X *weakly converges* to a measure μ if $\lim_n \int_X f d\mu_n = \int_X f d\mu$ for every bounded continuous function $f : X \to \mathbb{R}$. If X is a computable metric space, then the space $\mathcal{M}(X)$ of finite Borel measures on X is a computable metric space when equipped with the Prokhorov metric [9]. In turn, the Prokhorov metric induces the topology of weak convergence of measures on $\mathcal{M}(X)$ [3]. Weak convergence of measures is a useful tool in probability theory, as it can be used to create stable probability distributions and demonstrate convergence in distribution [4]. Moreover, weak convergence of measures underlies current research in computable measure theory and algorithmic randomness [2, 5, 6, 8].

We seek to establish a suitable framework in which to study the effective theory of weak convergence of measures. For instance, a weakly convergent sequence of measures need not have a computable limit even if the sequence is uniformly computable. In particular, any sequence of measures generated by iterating a computable measure through a cellular automaton is uniformly computable and converges weakly to a Δ^0_2-computable measure [7]. Since the conditions under which the weak limit of a computable sequence of measures is computable appear to not have been investigated, we are led to find a suitable effective definition of weak convergence of measures.

In this talk, we propose the following two effective definitions of weak convergence of measures in $\mathcal{M}(\mathbb{R})$.

Definition 1. We say $\{\mu_n\}_{n\in\mathbb{N}}$ *effectively weakly converges* to μ if for every bounded computable function $f : \mathbb{R} \to \mathbb{R}$, $\lim_n \int_{\mathbb{R}} f\ d\mu_n = \int_{\mathbb{R}} f\ d\mu$ and it is possible to compute an index of a modulus of convergence of $\{\int_{\mathbb{R}} f\ d\mu_n\}_{n\in\mathbb{N}}$ from an index of f and a bound $B \in \mathbb{N}$ on $|f|$.

Definition 2. We say $\{\mu_n\}_{n\in\mathbb{N}}$ *uniformly effectively weakly converges* to μ if it weakly converges to μ and there is a uniform procedure that for any bounded continuous function $f : \mathbb{R} \to \mathbb{R}$ computes a modulus of convergence of $\{\int_{\mathbb{R}} f\ d\mu_n\}_{n\in\mathbb{N}}$ from a name of f and a bound $B \in \mathbb{N}$ on $|f|$.

Note that *effective weak convergence* is not uniform and does not directly imply the classical definition. In contrast, *uniform effective weak convergence* is uniform and implies classical weak convergence. Nevertheless, the limits obtained from Definitions 1 and 2 are computable. Moreover, we obtain our first main result.

Theorem 1 ([10]). *Suppose $\{\mu_n\}_{n\in\mathbb{N}}$ is uniformly computable. The following are equivalent.*

(1) *$\{\mu_n\}_{n\in\mathbb{N}}$ is effectively weakly convergent.*
(2) *$\{\mu_n\}_{n\in\mathbb{N}}$ is uniformly effectively weakly convergent.*

In addition, we provide evidence that our definition of effective weak convergence in $\mathcal{M}(\mathbb{R})$ is the appropriate computable analogue to classical weak convergence in $\mathcal{M}(\mathbb{R})$. First, we need the following.

Definition 3. Suppose $\{a_n\}_{n\in\mathbb{N}}$ is a sequence of reals, and let $g :\subseteq \mathbb{Q} \to \mathbb{N}$.

(1) We say g *witnesses that* $\liminf_n a_n$ *is not smaller than* a if $\text{dom}(g)$ is the left Dedekind cut of a and if $r < a_n$ whenever $r \in \text{dom}(g)$ and $n \geq g(r)$.
(2) We say g *witnesses that* $\limsup_n a_n$ *is not larger than* a if $\text{dom}(g)$ is the right Dedekind cut of a and if $r > a_n$ whenever $r \in \text{dom}(g)$ and $n \geq g(r)$.

Our second main result is an effective version of the Portmanteau Theorem in $\mathcal{M}(\mathbb{R})$, a theorem originally due to Alexandroff [1] comprised of multiple equivalent definitions of weak convergence of measures. The theorem is stated as follows.

Theorem 2 (Effective Portmanteau Theorem, [10])**.** *Let* $\{\mu_n\}_{n\in\mathbb{N}}$ *be a uniformly computable sequence in* $\mathcal{M}(\mathbb{R})$*. The following are equivalent.*

(1) $\{\mu_n\}_{n\in\mathbb{N}}$ *effectively weakly converges to* μ*.*
(2) *From* $e, B \in \mathbb{N}$ *such that* e *indexes a bounded uniformly continuous function* $f : \mathbb{R} \to \mathbb{R}$ *with* $|f| \leq B$*, it is possible to compute a modulus of convergence of* $\{\int_{\mathbb{R}} f \; d\mu_n\}_{n\in\mathbb{N}}$*.*
(3) μ *is computable, and from an index of* $C \in \Pi^0_1(\mathbb{R})$ *it is possible to compute an index of a witness that* $\limsup_n \mu_n(C)$ *is not larger than* $\mu(C)$*.*
(4) μ *is computable, and from an index of* $U \in \Sigma^0_1(\mathbb{R})$ *it is possible to compute an index of a witness that* $\liminf_n \mu_n(U)$ *is not smaller than* $\mu(U)$*.*
(5) μ *is computable, and for every* μ*-almost decidable* A*,* $\lim_n \mu_n(A) = \mu(A)$ *and an index of a modulus of convergence of* $\{\mu_n(A)\}_{n\in\mathbb{N}}$ *can be computed from a* μ*-almost decidable index of* A*.*

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