Computable Multifunctions on Effectively Hausdorff Spaces

Matthias Schröder

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We continue our research on characterising realizable multifunctions (cf. [5]). Computable multifunctions play a prominent rôle in Computable Analysis [1, 7]. They are used as an appropriate substitute for single-valued functions that fail to be computable just for topological reasons. For example, the discontinuous and thus incomputable test "$x < y$" on the real numbers can often by replaced by the computable *finite precision test* $(x <_\varepsilon y)\colon \mathbb{R}\times\mathbb{R}_{>0}\times\mathbb{R} \rightrightarrows \{\texttt{tt},\texttt{ff}\}$ which allows both answers for inputs satisfying $y-\varepsilon < x < y$. (cf. [1]).

We introduce a new notion of an effectively Hausdorff space which is stronger than the current notion of a computable Hausdorff space proposed by A. Pauly in [3]. We then characterise computable multifunctions from computable metric spaces to effectively Hausdorff spaces. Inspired by this characterisation, we propose a stronger computability notion for multifunctions than the usual one and study some of its properties.

The usual notion of a computable multifunction

A multifunction F between admissibly represented spaces X,Y is a binary relation between the underlying sets of X and Y, written as $F\colon \mathsf{X} \rightrightarrows \mathsf{Y}$. The elements of $F[x] := \{y \in \mathsf{Y} \,|\, (x,y) \in F\}$ are viewed as the legitimate results for the input x. A multifunction F is called *computable* [6, 7], if there is a computable function $g\colon \mathbb{N}^\mathbb{N} \rightharpoonup \mathbb{N}^\mathbb{N}$ satisfying

$$\delta_\mathsf{Y}(g(p)) \in F[\delta_\mathsf{X}(p)] \quad \text{for every } p \in \mathrm{dom}(F\delta_\mathsf{X}).$$

Here δ_X and δ_Y denote the respective representations of X and Y.

Effectively Hausdorff spaces

It is known that the topology of every Hausdorff qcb-space contains a countably-based Hausdorff subtopology. We employ this fact in our definition of an effectively Hausdorff space.

Definition 1 We call a represented space X an *effectively Hausdorff space*, if there are two computable sequences $(\mathrm{u}_i)_i, (\mathrm{v}_i)_i$ of open subsets of X such that $\bigcup_i(\mathrm{u}_i\times\mathrm{v}_i) = \mathsf{X}^2\setminus\{(x,x)\,|\,x\in\mathsf{X}\}$ and the representation δ_X is computably admissible.

Any computable metric space is an example of an effectively Hausdorff space. Effectively Hausdorff spaces are computable Hausdorff spaces in the sense of [3], but not vice versa. Indeed, every effectively Hausdorff space is topologically Hausdorff, whereas computable Hausdorffness only entails sequential Hausdorffness (meaning that every convergent sequence has a unique limit). Effective Hausdorffness admits effectivisations of classical theorems from topology: for example the fact that compact Hausdorff spaces are regular can be effectivised by stating that any computably compact, effectively Hausdorff space is computably regular[1].

We apply effective Hausdorffness in the following characterisation theorem of computable multifunctions.

[1] See [3] for the definition of computable compactness and [4] for computable regularity.

Theorem 2 *Let X be a computable metric space, and let Y be an effectively Hausdorff space. Then a total multifunction $F\colon \mathsf{X} \rightrightarrows \mathsf{X}$ is computable iff there are two computable functions $h_+\colon \mathsf{X} \to \mathcal{A}_+(\mathsf{Y})$ and $h_b\colon \mathsf{X} \to \mathcal{K}_-(\mathsf{Y})$ satisfying $\emptyset \neq h_+(x) \subseteq F[x] \cap h_b(x)$ for all $x \in \mathsf{X}$.*

Here $\mathcal{A}_+(\mathsf{Y})$ denotes the represented space of non-empty closed subsets of Y with the positive (overt) representation and $\mathcal{K}_-(\mathsf{Y})$ the represented space of compact subsets of Y with the negative representation (cf. [2, 3, 6]). Note that $h_+(x)$ is in fact compact by being a closed subset of a compact set.

Strongly computable multifunctions

The only-if-part of Theorem 2 only holds for metrisable spaces. We now introduce a refined notion of computability for multifunctions which is based on computability of set-valued functions and works appropriately at least for multifunctions between effectively Hausdorff spaces.

Definition 3 We call a total multifunction F *strongly computable*, if there are two computable functions $h_+\colon \mathsf{X} \to \mathcal{A}_+(\mathsf{Y})$, $h_-\colon \mathsf{X} \to \mathcal{K}_-(\mathsf{Y})$ satisfying $\emptyset \neq h_+(x) \subseteq F[x] \subseteq h_-(x)$ for all $x \in \mathsf{X}$.

This computability notion chimes with a composition $\diamond$ for multifunctions defined by

$$G \diamond F[x] := \mathrm{Cls} \bigcup \big\{ G[y] \,\big|\, y \in F[x] \big\}.$$

If the involved spaces are effectively Hausdorff, then $\diamond$ preserves strong computability; moreover, any strongly computable multifunction is computable in the usual sense and computable total functions form strongly computable multifunctions. We obtain:

Proposition 4 *Effectively Hausdorff spaces together with strongly computable multifunctions having compact images form a category (*which we denote by $\mathsf{EffHausMulti}$*). It has equalisers and finite coproducts.*

Discussion

Any (in the usual sense) computable multifunction from a computable metric space X to an effectively Hausdorff space has a tightening which is strongly computable. This is not true, if X is not metrisable. The category $\mathsf{EffHausMulti}$ is not cartesian, it only has 'weak' products. It is not known whether effective Hausdorffness is necessary for Theorem 2 or Proposition 4.

References

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