Uniform Complexity of Solving Partial Differential Equations and Exact Real Computation*

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We present some recent results on the problem of computing solutions of partial differential equations in the sense of exact real computation, that is with guaranteed *arbitrary* output precision. More precisely, we consider Cauchy-Kovalevskaya type systems of linear partial differential equations with variable coefficients and given initial values:

$$\partial_t \boldsymbol{u}(\boldsymbol{x},t) = f_1(\boldsymbol{x})\partial_1 \boldsymbol{u} \ + \cdots + \ f_d(\boldsymbol{x})\partial_d \boldsymbol{u} \qquad \boldsymbol{u}(\boldsymbol{x},0) \equiv v(\vec{x}) \ . \tag{1}$$

In [KPSZ21, KSZ19] it is shown that the finite difference approach, adapted to the ERC paradigm, is in the complexity class PSPACE, w.r.t. n, with fixed polynomial time computable initial functions and matrix coefficients. It can be improved at best to $\#\mathsf{P}$ in the constant periodic case even for analytic polynomial time computable initial functions, since so are the exponential-size matrix powering and inner product. The power series approach on the other hand does yield a polynomial time algorithm w.r.t. n for fixed polynomial time computable analytic functions [KSZ19].

In recent work [SSTZ21] we improve this nonuniform result to a uniform algorithm, where the right-hand side functions $f_1, \ldots, f_d$ and the initial value v are not fixed but given, together with certain parameters that quantitatively capture their convergence behaviour. Following ideas from [KMRZ15, KST18] we encode analytic functions by their (multivariate) power series together with integer parameters $L, M \in \mathbb{N}$ (called coefficient bounds) bounding the speed of convergence of the series and use this representation as basis for our algorithms:

Theorem 1 *Fix $d \in \mathbb{N}$ and consider the solution operator that maps any analytic right-hand sides $f_1, \ldots, f_d : [-1;1]^d \to \mathbb{C}^{d' \times d'}$ and initial condition $v : [-1;1]^d \to \mathbb{C}^{d'}$ and 'small' enough $t \in \mathbb{C}$ to the solution $u = u(t, \cdot)$ of (1).*
This operator is computable in parameterized time polynomial in $n + L + \log M$, where $v, f_1, \ldots, f_d$ are given via their (componentwise) Taylor expansions around $\vec{0}$ as well as integers (M, L) as coefficient bounds to $v, f_1, \ldots, f_d$ componentwise.

We further extend the results to some other examples of PDEs that are not covered by the Cauchy-Kovalevskaya theorem such as the Heat- and Schrödinger equations. We also discuss implementations in exact real arithmetic frameworks such as `iRRAM` [Mül01] and possible connections to formal verification in proof assistants.

The talk is largely based on joint work with Svetlana Selivanova, Florian Steinberg and Martin Ziegler.

References

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