An effective completeness theorem for continuous logic and computable presentations

Eighteenth International Conference
on Computability and Complexity in Analysis
July 26, 2021

Caleb Camrud

Motivation

Which model-theoretic results have effective counterparts?

Motivation

Which model-theoretic results have effective counterparts?

A wide range of results have been proven in this area with respect to classical logic, surveyed by Harizanov in *Pure Computable Model Theory*.

Motivation

However, classical model theory has limitations in its application to non-discrete metric structures.

Motivation

However, classical model theory has limitations in its application to non-discrete metric structures.

Because of this, Ben Yaacov *et al.* developed a model theory for metric structures using continuous first-order logic.

Motivation

However, classical model theory has limitations in its application to non-discrete metric structures.

Because of this, Ben Yaacov *et al.* developed a model theory for metric structures using continuous first-order logic.

Ben Yaacov and Pedersen soon proved a completeness result for this logic.

Motivation

Then in *Metric Structures and Probablistic Computation*, Calvert proved a version of an effective completeness theorem for continuous logic:

Motivation

Then in *Metric Structures and Probablistic Computation*, Calvert proved a version of an effective completeness theorem for continuous logic:

Theorem

Let T be a complete, decidable, continuous first-order theory. Then there is a probablistically decidable, continuous weak structure $\mathfrak{M}$ such that $\mathfrak{M} \models T$.

Motivation

In the recent decade, however, computable presentations have become the norm for the study of effectivity on metric structures (see works by McNicholl, Melnikov, Franklin, *etc.*).

Motivation

In the recent decade, however, computable presentations have become the norm for the study of effectivity on metric structures (see works by McNicholl, Melnikov, Franklin, *etc.*).

We therefore sought to answer the following.

Motivation

In the recent decade, however, computable presentations have become the norm for the study of effectivity on metric structures (see works by McNicholl, Melnikov, Franklin, *etc.*).

We therefore sought to answer the following.

Is there an effective completeness theorem for continuous logic and computable presentations? (As opposed to probabilistically decidable structures.)

Motivation

The answer to this question is positive.

Motivation

The answer to this question is positive.

Theorem

There is an effective procedure which uniformly, given an index of a complete, decidable L-theory T, produces an index of a computable presentation of an L^+-structure $\mathfrak{M}$ such that $\mathfrak{M} \models T$.

Scope of presentation

1. Continuous logic

Scope of presentation

1. Continuous logic
2. Metric structures

Scope of presentation

1. Continuous logic
2. Metric structures
3. Computable presentations

Scope of presentation

1. Continuous logic
2. Metric structures
3. Computable presentations
4. Classical completeness results

Scope of presentation

1. Continuous logic
2. Metric structures
3. Computable presentations
4. Classical completeness results
5. Effective extensions of decidable theories

Scope of presentation

1. Continuous logic
2. Metric structures
3. Computable presentations
4. Classical completeness results
5. Effective extensions of decidable theories
6. An effective completeness theorem

The language of continuous logic

A *signature* is a quintuple $L = (\mathcal{P}, \mathcal{F}, \mathcal{C}, \eta, \Delta)$ such that

The language of continuous logic

A *signature* is a quintuple $L = (\mathcal{P}, \mathcal{F}, \mathcal{C}, \eta, \Delta)$ such that

- $\mathcal{P}$, $\mathcal{F}$, and $\mathcal{C}$ are disjoint,

The language of continuous logic

A *signature* is a quintuple $L = (\mathcal{P}, \mathcal{F}, \mathcal{C}, \eta, \Delta)$ such that

- $\mathcal{P}$, $\mathcal{F}$, and $\mathcal{C}$ are disjoint,
- $\mathcal{P}$ is a set of *predicate symbols*, including a distinguished predicate symbol δ,

The language of continuous logic

A *signature* is a quintuple $L = (\mathcal{P}, \mathcal{F}, \mathcal{C}, \eta, \Delta)$ such that

- $\mathcal{P}$, $\mathcal{F}$, and $\mathcal{C}$ are disjoint,
- $\mathcal{P}$ is a set of *predicate symbols*, including a distinguished predicate symbol δ,
- $\mathcal{F}$ is a set of *function symbols*,

The language of continuous logic

A *signature* is a quintuple $L = (\mathcal{P}, \mathcal{F}, \mathcal{C}, \eta, \Delta)$ such that

- $\mathcal{P}$, $\mathcal{F}$, and $\mathcal{C}$ are disjoint,
- $\mathcal{P}$ is a set of *predicate symbols*, including a distinguished predicate symbol δ,
- $\mathcal{F}$ is a set of *function symbols*,
- $\mathcal{C}$ is a nonempty set of *constant symbols*,

The language of continuous logic

A *signature* is a quintuple $L = (\mathcal{P}, \mathcal{F}, \mathcal{C}, \eta, \Delta)$ such that

- $\mathcal{P}$, $\mathcal{F}$, and $\mathcal{C}$ are disjoint,
- $\mathcal{P}$ is a set of *predicate symbols*, including a distinguished predicate symbol δ,
- $\mathcal{F}$ is a set of *function symbols*,
- $\mathcal{C}$ is a nonempty set of *constant symbols*,
- η assigns to each predicate or function symbol a natural number, with $\eta(\delta) = 2$, and

The language of continuous logic

A *signature* is a quintuple $L = (\mathcal{P}, \mathcal{F}, \mathcal{C}, \eta, \Delta)$ such that

- $\mathcal{P}$, $\mathcal{F}$, and $\mathcal{C}$ are disjoint,
- $\mathcal{P}$ is a set of *predicate symbols*, including a distinguished predicate symbol δ,
- $\mathcal{F}$ is a set of *function symbols*,
- $\mathcal{C}$ is a nonempty set of *constant symbols*,
- η assigns to each predicate or function symbol a natural number, with $\eta(\delta) = 2$, and
- Δ assigns to each predicate or function symbol F, a map $\Delta_F : \mathbb{N} \to \mathbb{N}$, with $\Delta_\delta = \mathrm{id}_{\mathbb{N}}$.

The language of continuous logic

By a *variable symbol*, we mean a symbol from a distinguished infinite set $\{x, y, z, ...\}$ which is disjoint from $\mathcal{P}$, $\mathcal{F}$, and $\mathcal{C}$.

The language of continuous logic

By a *variable symbol*, we mean a symbol from a distinguished infinite set $\{x, y, z, ...\}$ which is disjoint from $\mathcal{P}$, $\mathcal{F}$, and $\mathcal{C}$.

By a *quantifier*, we mean a symbol from the set $\{\sup, \inf\}$.

The language of continuous logic

By a *variable symbol*, we mean a symbol from a distinguished infinite set $\{x, y, z, ...\}$ which is disjoint from $\mathcal{P}$, $\mathcal{F}$, and $\mathcal{C}$.

By a *quantifier*, we mean a symbol from the set $\{\sup, \inf\}$.

And by a *connective*, we mean a symbol from the set $\{\neg, \frac{1}{2}, \dot{-}\}$.

The language of continuous logic

Given a signature L, the *terms of L* are defined as in classical logic, *i.e.*

The language of continuous logic

Given a signature L, the *terms of L* are defined as in classical logic, *i.e.*

1. Every constant and variable symbol is a term of L.

The language of continuous logic

Given a signature L, the *terms of L* are defined as in classical logic, *i.e.*

1. Every constant and variable symbol is a term of L.
2. If f is an N-ary function symbol of L and $t_0, \ldots, t_{N-1}$ are terms of L, then $f(t_0, \ldots, t_{N-1})$ is a term of L.

The language of continuous logic

The *well-formed formulas of L* (*L-wffs*, for short) are also defined as in classical logic, but with continuous connectives and quantifiers, *i.e.*

The language of continuous logic

The *well-formed formulas of L* (*L-wffs*, for short) are also defined as in classical logic, but with continuous connectives and quantifiers, *i.e.*

1. If P is an N-ary predicate symbol of L and $t_0, \ldots, t_{N-1}$ are terms of L, then $P(t_0, \ldots, t_{N-1})$ is an L-wff.

The language of continuous logic

The *well-formed formulas of L* (*L-wffs*, for short) are also defined as in classical logic, but with continuous connectives and quantifiers, *i.e.*

1. If P is an N-ary predicate symbol of L and $t_0, \ldots, t_{N-1}$ are terms of L, then $P(t_0, \ldots, t_{N-1})$ is an L-wff.
2. If φ is an L-wff, then both $\neg\varphi$ and $\frac{1}{2}\varphi$ are L-wffs.

The language of continuous logic

The *well-formed formulas of L* (*L-wffs*, for short) are also defined as in classical logic, but with continuous connectives and quantifiers, *i.e.*

1. If P is an N-ary predicate symbol of L and $t_0, \ldots, t_{N-1}$ are terms of L, then $P(t_0, \ldots, t_{N-1})$ is an L-wff.
2. If φ is an L-wff, then both $\neg\varphi$ and $\frac{1}{2}\varphi$ are L-wffs.
3. If φ and ψ are L-wffs, then $\varphi \dot{-} \psi$ is an L-wff.

The language of continuous logic

The *well-formed formulas of L* (*L-wffs*, for short) are also defined as in classical logic, but with continuous connectives and quantifiers, *i.e.*

1. If P is an N-ary predicate symbol of L and $t_0, \ldots, t_{N-1}$ are terms of L, then $P(t_0, \ldots, t_{N-1})$ is an L-wff.
2. If φ is an L-wff, then both $\neg\varphi$ and $\frac{1}{2}\varphi$ are L-wffs.
3. If φ and ψ are L-wffs, then $\varphi \dot{-} \psi$ is an L-wff.
4. If φ is an L-wff and x is a variable symbol, then both $\sup_x \varphi$ and $\inf_x \varphi$ are L-wffs.

The language of continuous logic

Free and *bound* variables are defined in the standard manner. An *L-sentence* is an L-wff which includes no free variables.

The language of continuous logic

Free and *bound* variables are defined in the standard manner. An *L-sentence* is an L-wff which includes no free variables.

Though we will be proving an effective completeness result, the actual axioms and rule of modus ponens for continuous logic are not important for this paper. All our results are obtainable by means of the completeness results proven in (Ben Yaacov and Pedersen, 2010).

The language of continuous logic

Free and *bound* variables are defined in the standard manner. An *L-sentence* is an L-wff which includes no free variables.

Though we will be proving an effective completeness result, the actual axioms and rule of modus ponens for continuous logic are not important for this paper. All our results are obtainable by means of the completeness results proven in (Ben Yaacov and Pedersen, 2010).

An *L-theory* is then a consistent set of L-wffs.

Metric structures

Suppose $(|\mathfrak{M}|, d)$ and $(|\mathfrak{M}'|, d')$ are pseudometric spaces, and let $f : |\mathfrak{M}| \to |\mathfrak{M}'|$.

Metric structures

Suppose $(|\mathfrak{M}|, d)$ and $(|\mathfrak{M}'|, d')$ are pseudometric spaces, and let $f : |\mathfrak{M}| \to |\mathfrak{M}'|$.

A map $\Delta_f : \mathbb{N} \to \mathbb{N}$ is called a *modulus of uniform continuity for f* if $d'(f(a), f(b)) \leqslant 2^{-k}$ implies that $d(a, b) \leqslant 2^{-\Delta_f(k)}$.

Metric structures

Suppose $(|\mathfrak{M}|, d)$ and $(|\mathfrak{M}'|, d')$ are pseudometric spaces, and let $f : |\mathfrak{M}| \to |\mathfrak{M}'|$.

A map $\Delta_f : \mathbb{N} \to \mathbb{N}$ is called a *modulus of uniform continuity for f* if $d'(f(a), f(b)) \leqslant 2^{-k}$ implies that $d(a, b) \leqslant 2^{-\Delta_f(k)}$.

When f is not unary, the definition extends if it holds in every argument.

Metric structures

Fix a pseudometric space $(|\mathfrak{M}|, d)$ of diameter 1.

Metric structures

Fix a pseudometric space $(|\mathfrak{M}|, d)$ of diameter 1. We say that a map $\cdot^{\mathfrak{M}}$ with domain $\mathcal{P} \cup \mathcal{F} \cup \mathcal{C}$ is an *L-interpretation over* $(|\mathfrak{M}|, d)$ if each of the following hold.

Metric structures

Fix a pseudometric space $(|\mathfrak{M}|, d)$ of diameter 1. We say that a map $\cdot^{\mathfrak{M}}$ with domain $\mathcal{P} \cup \mathcal{F} \cup \mathcal{C}$ is an *L-interpretation over* $(|\mathfrak{M}|, d)$ if each of the following hold.

1. $\delta^{\mathfrak{M}} = d$.

Metric structures

Fix a pseudometric space $(|\mathfrak{M}|, d)$ of diameter 1. We say that a map $\cdot^{\mathfrak{M}}$ with domain $\mathcal{P} \cup \mathcal{F} \cup \mathcal{C}$ is an *L-interpretation over* $(|\mathfrak{M}|, d)$ if each of the following hold.

1. $\delta^{\mathfrak{M}} = d$.
2. If P is a predicate symbol of L, then $P^{\mathfrak{M}} : |\mathfrak{M}|^{\eta(P)} \rightarrow [0, 1]$ and Δ_P is a modulus of uniform continuity for P.

Metric structures

Fix a pseudometric space $(|\mathfrak{M}|, d)$ of diameter 1. We say that a map $\cdot^{\mathfrak{M}}$ with domain $\mathcal{P} \cup \mathcal{F} \cup \mathcal{C}$ is an *L-interpretation over* $(|\mathfrak{M}|, d)$ if each of the following hold.

1. $\delta^{\mathfrak{M}} = d$.
2. If P is a predicate symbol of L, then $P^{\mathfrak{M}} : |\mathfrak{M}|^{\eta(P)} \to [0, 1]$ and Δ_P is a modulus of uniform continuity for P.
3. If f is a function symbol of L, then $f^{\mathfrak{M}} : |\mathfrak{M}|^{\eta(f)} \to |\mathfrak{M}|$ and Δ_f is a modulus of uniform continuity for f.

Metric structures

Fix a pseudometric space $(|\mathfrak{M}|, d)$ of diameter 1. We say that a map $\cdot^{\mathfrak{M}}$ with domain $\mathcal{P} \cup \mathcal{F} \cup \mathcal{C}$ is an *L-interpretation over* $(|\mathfrak{M}|, d)$ if each of the following hold.

1. $\delta^{\mathfrak{M}} = d$.
2. If P is a predicate symbol of L, then $P^{\mathfrak{M}} : |\mathfrak{M}|^{\eta(P)} \to [0, 1]$ and Δ_P is a modulus of uniform continuity for P.
3. If f is a function symbol of L, then $f^{\mathfrak{M}} : |\mathfrak{M}|^{\eta(f)} \to |\mathfrak{M}|$ and Δ_f is a modulus of uniform continuity for f.
4. If c is a constant symbol of L, then $c^{\mathfrak{M}} \in |\mathfrak{M}|$.

Metric structures

When $(|\mathfrak{M}|, d)$ is a pseudometric space of diameter 1 and $\cdot^{\mathfrak{M}}$ is an L-interpretation over $(|\mathfrak{M}|, d)$, we say that

$$\mathfrak{M} = \left(|\mathfrak{M}|, d, \{P^{\mathfrak{M}} : P \in \mathcal{P}\}, \{f^{\mathfrak{M}} : f \in \mathcal{F}\}, \{c^{\mathfrak{M}} : c \in \mathcal{C}\}\right)$$

is an *L-pre-structure*.

Metric structures

When $(|\mathfrak{M}|, d)$ is a pseudometric space of diameter 1 and $\cdot^{\mathfrak{M}}$ is an L-interpretation over $(|\mathfrak{M}|, d)$, we say that

$$\mathfrak{M} = \left(|\mathfrak{M}|, d, \{P^{\mathfrak{M}} : P \in \mathcal{P}\}, \{f^{\mathfrak{M}} : f \in \mathcal{F}\}, \{c^{\mathfrak{M}} : c \in \mathcal{C}\}\right)$$

is an *L-pre-structure*.

If, moreover, $(|\mathfrak{M}|, d)$ is a complete metric space, then $\mathfrak{M}$ is an *L-structure*.

Metric structures

When $(|\mathfrak{M}|, d)$ is a pseudometric space of diameter 1 and $\cdot^{\mathfrak{M}}$ is an L-interpretation over $(|\mathfrak{M}|, d)$, we say that

$$\mathfrak{M} = \left(|\mathfrak{M}|, d, \{P^{\mathfrak{M}} : P \in \mathcal{P}\}, \{f^{\mathfrak{M}} : f \in \mathcal{F}\}, \{c^{\mathfrak{M}} : c \in \mathcal{C}\}\right)$$

is an *L-pre-structure*.

If, moreover, $(|\mathfrak{M}|, d)$ is a complete metric space, then $\mathfrak{M}$ is an *L-structure*.

In either case, $\{P^{\mathfrak{M}} : P \in \mathcal{P}\}$ is called the *set of predicates of* $\mathfrak{M}$, $\{f^{\mathfrak{M}} : f \in \mathcal{F}\}$ is called the *set of functions of* $\mathfrak{M}$, and $\{c^{\mathfrak{M}} : c \in \mathcal{C}\}$ is called the *set of distinguished points of* $\mathfrak{M}$.

Metric structures

Given an L-sentence φ and an L-pre-structure $\mathfrak{M}$, define the *value of* φ *in* $\mathfrak{M}$, denoted $\varphi^{\mathfrak{M}}$, recursively as follows.

Metric structures

Given an L-sentence φ and an L-pre-structure $\mathfrak{M}$, define the *value of* φ *in* $\mathfrak{M}$, denoted $\varphi^{\mathfrak{M}}$, recursively as follows.

1. $\left(P(t_0, ..., t_{N-1})\right)^{\mathfrak{M}} := P^{\mathfrak{M}}(t_0^{\mathfrak{M}}, ..., t_{N-1}^{\mathfrak{M}})$.

Metric structures

Given an L-sentence φ and an L-pre-structure $\mathfrak{M}$, define the *value of* φ *in* $\mathfrak{M}$, denoted $\varphi^{\mathfrak{M}}$, recursively as follows.

1. $\big(P(t_0, \ldots, t_{N-1})\big)^{\mathfrak{M}} := P^{\mathfrak{M}}(t_0^{\mathfrak{M}}, \ldots, t_{N-1}^{\mathfrak{M}})$.
2. $(\neg\varphi)^{\mathfrak{M}} := 1 - \varphi^{\mathfrak{M}}$.

Metric structures

Given an L-sentence φ and an L-pre-structure $\mathfrak{M}$, define the *value of* φ *in* $\mathfrak{M}$, denoted $\varphi^{\mathfrak{M}}$, recursively as follows.

1. $\left(P(t_0, \ldots, t_{N-1})\right)^{\mathfrak{M}} := P^{\mathfrak{M}}(t_0^{\mathfrak{M}}, \ldots, t_{N-1}^{\mathfrak{M}})$.
2. $(\neg\varphi)^{\mathfrak{M}} := 1 - \varphi^{\mathfrak{M}}$.
3. $\left(\frac{1}{2}\varphi\right)^{\mathfrak{M}} := \frac{1}{2}\ \varphi^{\mathfrak{M}}$.

Metric structures

Given an L-sentence φ and an L-pre-structure $\mathfrak{M}$, define the *value of* φ *in* $\mathfrak{M}$, denoted $\varphi^{\mathfrak{M}}$, recursively as follows.

1. $\left(P(t_0, \ldots, t_{N-1})\right)^{\mathfrak{M}} := P^{\mathfrak{M}}(t_0^{\mathfrak{M}}, \ldots, t_{N-1}^{\mathfrak{M}})$.
2. $(\neg\varphi)^{\mathfrak{M}} := 1 - \varphi^{\mathfrak{M}}$.
3. $\left(\frac{1}{2}\varphi\right)^{\mathfrak{M}} := \frac{1}{2}\ \varphi^{\mathfrak{M}}$.
4. $(\varphi \dot{-} \psi)^{\mathfrak{M}} := \max\{0, \varphi^{\mathfrak{M}} - \psi^{\mathfrak{M}}\}$.

Metric structures

Given an L-sentence φ and an L-pre-structure $\mathfrak{M}$, define the *value of* φ *in* $\mathfrak{M}$, denoted $\varphi^{\mathfrak{M}}$, recursively as follows.

1. $\big(P(t_0, \dots, t_{N-1})\big)^{\mathfrak{M}} := P^{\mathfrak{M}}(t_0^{\mathfrak{M}}, \dots, t_{N-1}^{\mathfrak{M}})$.
2. $(\neg\varphi)^{\mathfrak{M}} := 1 - \varphi^{\mathfrak{M}}$.
3. $\left(\frac{1}{2}\varphi\right)^{\mathfrak{M}} := \frac{1}{2}\,\varphi^{\mathfrak{M}}$.
4. $(\varphi \dot{-} \psi)^{\mathfrak{M}} := \max\{0, \varphi^{\mathfrak{M}} - \psi^{\mathfrak{M}}\}$.
5. $\big(\sup_x \varphi(x)\big)^{\mathfrak{M}} := \sup_{a \in |\mathfrak{M}|} \varphi^{\mathfrak{M}}(a)$.

Metric structures

Given an L-sentence φ and an L-pre-structure $\mathfrak{M}$, define the *value of φ in $\mathfrak{M}$*, denoted $\varphi^{\mathfrak{M}}$, recursively as follows.

1. $\big(P(t_0, \ldots, t_{N-1})\big)^{\mathfrak{M}} := P^{\mathfrak{M}}(t_0^{\mathfrak{M}}, \ldots, t_{N-1}^{\mathfrak{M}})$.
2. $(\neg\varphi)^{\mathfrak{M}} := 1 - \varphi^{\mathfrak{M}}$.
3. $\left(\frac{1}{2}\varphi\right)^{\mathfrak{M}} := \frac{1}{2}\ \varphi^{\mathfrak{M}}$.
4. $(\varphi \dot{-} \psi)^{\mathfrak{M}} := \max\{0, \varphi^{\mathfrak{M}} - \psi^{\mathfrak{M}}\}$.
5. $\big(\sup_x \varphi(x)\big)^{\mathfrak{M}} := \sup_{a \in |\mathfrak{M}|} \varphi^{\mathfrak{M}}(a)$.
6. $\big(\inf_x \varphi(x)\big)^{\mathfrak{M}} := \inf_{a \in |\mathfrak{M}|} \varphi^{\mathfrak{M}}(a)$.

Metric structures

When $\varphi^{\mathfrak{M}} = 0$, we say that $\mathfrak{M}$ *satisfies* φ, and write $\mathfrak{M} \models \varphi$.

Metric structures

When $\varphi^{\mathfrak{M}} = 0$, we say that $\mathfrak{M}$ *satisfies* φ, and write $\mathfrak{M} \models \varphi$.

To define the value of any L-wff in a given L-pre-structure $\mathfrak{M}$, first define an *assignment* on $\mathfrak{M}$ as $\sigma : \{x, y, z, ...\} \to |\mathfrak{M}|$.

Metric structures

When $\varphi^{\mathfrak{M}} = 0$, we say that $\mathfrak{M}$ *satisfies* φ, and write $\mathfrak{M} \models \varphi$.

To define the value of any L-wff in a given L-pre-structure $\mathfrak{M}$, first define an *assignment* on $\mathfrak{M}$ as $\sigma : \{x, y, z, ...\} \to |\mathfrak{M}|$. Then define $\varphi^{\mathfrak{M},\sigma}(x, y, ...) := \varphi^{\mathfrak{M}}(\sigma(x), \sigma(y), ...)$.

Metric structures

When $\varphi^{\mathfrak{M}} = 0$, we say that $\mathfrak{M}$ *satisfies* φ, and write $\mathfrak{M} \models \varphi$.

To define the value of any L-wff in a given L-pre-structure $\mathfrak{M}$, first define an *assignment* on $\mathfrak{M}$ as $\sigma : \{x, y, z, ...\} \to |\mathfrak{M}|$. Then define $\varphi^{\mathfrak{M},\sigma}(x, y, ...) := \varphi^{\mathfrak{M}}(\sigma(x), \sigma(y), ...)$.

When $\varphi^{\mathfrak{M},\sigma} = 0$, we say that $\mathfrak{M}$ *satisfies* φ *under assignment* σ, and write $\mathfrak{M}, \sigma \models \varphi$.

Computable presentations

A real number r is *computable* if there is an effective procedure which, given $k \in \mathbb{N}$, produces a rational number q such that $|r - q| < 2^{-k}$.

Computable presentations

Given an L-structure $\mathfrak{M}$ and $A \subseteq |\mathfrak{M}|$, define the *algebra generated by* A to be the smallest subset of $|\mathfrak{M}|$ containing A that is closed under every function of $\mathfrak{M}$.

Computable presentations

Given an L-structure $\mathfrak{M}$ and $A \subseteq |\mathfrak{M}|$, define the *algebra generated by A* to be the smallest subset of $|\mathfrak{M}|$ containing A that is closed under every function of $\mathfrak{M}$.

A pair $(\mathfrak{M}, g)$ is called a *presentation* of $\mathfrak{M}$ if $g : \mathbb{N} \to |\mathfrak{M}|$ is a map such that the algebra generated by $\mathrm{ran}(g)$ is dense.

Computable presentations

Given an L-structure $\mathfrak{M}$ and $A \subseteq |\mathfrak{M}|$, define the *algebra generated by A* to be the smallest subset of $|\mathfrak{M}|$ containing A that is closed under every function of $\mathfrak{M}$.

A pair $(\mathfrak{M}, g)$ is called a *presentation* of $\mathfrak{M}$ if $g : \mathbb{N} \to |\mathfrak{M}|$ is a map such that the algebra generated by $\mathrm{ran}(g)$ is dense.

Every point in $\mathrm{ran}(g)$ is called a *distinguished point* of the presentation, and each point in the algebra generated by the distinguished points is called a *rational point* of the presentation.

Computable presentations

A presentation $(\mathfrak{M}, g)$ is *computable* if the predicates of $\mathfrak{M}$ are uniformly computable on the rational points of $(\mathfrak{M}, g)$.

Computable presentations

A presentation $(\mathfrak{M}, g)$ is *computable* if the predicates of $\mathfrak{M}$ are uniformly computable on the rational points of $(\mathfrak{M}, g)$.

Since the metric is a binary predicate on $\mathfrak{M}$, this entails that the distance between any two rational points is uniformly computable.

Computable presentations

A presentation $(\mathfrak{M}, g)$ is *computable* if the predicates of $\mathfrak{M}$ are uniformly computable on the rational points of $(\mathfrak{M}, g)$.

Since the metric is a binary predicate on $\mathfrak{M}$, this entails that the distance between any two rational points is uniformly computable.

An *index* of a computable presentation $(\mathfrak{M}, g)$ is a code of a Turing machine that computes the predicates of $\mathfrak{M}$ on the rational points of $(\mathfrak{M}, g)$.

Dyadic sentences

Given a signature L, define the *dyadic L-sentences*, denoted Dyad_L, as the smallest set which contains $\delta(c, c)$ for every constant symbol c of L, and which is closed under $\neg$, $\frac{1}{2}$, and $\dot{-}$.

Dyadic sentences

Given a signature L, define the *dyadic L-sentences*, denoted Dyad_L, as the smallest set which contains $\delta(c, c)$ for every constant symbol c of L, and which is closed under $\neg$, $\frac{1}{2}$, and $\dot{-}$.

As noted in (Ben Yaacov and Pedersen, 2010), for every dyadic number $r \in [0, 1]$, there is a dyadic L-sentence $\widehat{r}$ such that for every L-structure $\mathfrak{M}$, $(\widehat{r})^{\mathfrak{M}} = r$.

Dyadic sentences

Given a signature L, define the *dyadic L-sentences*, denoted Dyad_L, as the smallest set which contains $\delta(c, c)$ for every constant symbol c of L, and which is closed under $\neg$, $\frac{1}{2}$, and $\dot{-}$.

As noted in (Ben Yaacov and Pedersen, 2010), for every dyadic number $r \in [0, 1]$, there is a dyadic L-sentence $\widehat{r}$ such that for every L-structure $\mathfrak{M}$, $(\widehat{r})^{\mathfrak{M}} = r$.

Because of this, Ben Yaacov and Pedersen refrain from distinguishing $r \in [0, 1]$ and any representative of the set of dyadic L-sentences which are universally interpreted as r.

Dyadic sentences

Given a signature L, define the *dyadic L-sentences*, denoted Dyad_L, as the smallest set which contains $\delta(c, c)$ for every constant symbol c of L, and which is closed under $\neg$, $\frac{1}{2}$, and $\dot{-}$.

As noted in (Ben Yaacov and Pedersen, 2010), for every dyadic number $r \in [0, 1]$, there is a dyadic L-sentence $\widehat{r}$ such that for every L-structure $\mathfrak{M}$, $(\widehat{r})^{\mathfrak{M}} = r$.

Because of this, Ben Yaacov and Pedersen refrain from distinguishing $r \in [0, 1]$ and any representative of the set of dyadic L-sentences which are universally interpreted as r. To be precise about notions of effectivity, however, we will need to preserve this distinction.

Dyadic sentences

We say that signature L is *effectively encoded* if the function, predicate, constant, and variable symbols are effectively encoded as natural numbers and the arity and moduli of uniform continuity maps are computable.

Dyadic sentences

We say that signature L is *effectively encoded* if the function, predicate, constant, and variable symbols are effectively encoded as natural numbers and the arity and moduli of uniform continuity maps are computable.

If L is effectively encoded, it follows that the set of L-wffs, the set of L-sentences, and the set of dyadic L-sentences are all computable.

Dyadic sentences

Proposition

Let L be an effectively encoded signature L. Then there is a computable map which uniformly, given a dyadic number $r \in [0, 1]$, produces a dyadic L-sentence $\widehat{r}$ such that for every L-structure $\mathfrak{M}$, $(\widehat{r})^{\mathfrak{M}} = r$.

Dyadic sentences

When L is an effectively encoded signature, we restrict Dyad_L to the range of the computable map constructed in the above proposition.

Dyadic sentences

When L is an effectively encoded signature, we restrict Dyad_L to the range of the computable map constructed in the above proposition. In this way, the map becomes a bijection.

Dyadic sentences

When L is an effectively encoded signature, we restrict Dyad_L to the range of the computable map constructed in the above proposition. In this way, the map becomes a bijection.

Since we use $\widehat{r}$ to denote the dyadic L-sentence which is interpreted as r in every L-structure $\mathfrak{M}$, we use $\check{p}$ to denote the opposite.

Dyadic sentences

When L is an effectively encoded signature, we restrict Dyad_L to the range of the computable map constructed in the above proposition. In this way, the map becomes a bijection.

Since we use $\widehat{r}$ to denote the dyadic L-sentence which is interpreted as r in every L-structure $\mathfrak{M}$, we use $\breve{p}$ to denote the opposite. That is, given a dyadic L-sentence p, $\breve{p}$ is the unique dyadic number in $[0, 1]$ such that for every L-structure $\mathfrak{M}$, $p^{\mathfrak{M}} = \breve{p}$.

Classical completeness results for continuous logic

Let L be a signature. Define the *Henkin extended signature of* L, denoted L^+, to be the smallest signature that extends L and that, for every combination of L^+-wff φ, variable symbol x, and $p, q \in \mathrm{Dyad}_{L^+}$, contains a unique constant symbol $c_{\varphi,x,p,q}$.

Classical completeness results for continuous logic

Let L be a signature. Define the *Henkin extended signature of L*, denoted L^+, to be the smallest signature that extends L and that, for every combination of L^+-wff φ, variable symbol x, and $p, q \in \mathrm{Dyad}_{L^+}$, contains a unique constant symbol $c_{\varphi,x,p,q}$.

When Γ is an L^+-theory, we say it is *Henkin complete* if for every L-wff φ, every variable symbol x, and every $p, q \in \mathrm{Dyad}_{L^+}$,

$$(\sup_x \varphi \dot{-} q) \wedge (p \dot{-} \varphi[c_{\varphi,x,p,q}/x]) \in \Gamma.$$

Classical completeness results for continuous logic

We say that Γ is *maximally consistent* if Γ is consistent and for every pair of L^+-wffs φ and ψ, we have the following.

Classical completeness results for continuous logic

We say that Γ is *maximally consistent* if Γ is consistent and for every pair of L^+-wffs φ and ψ, we have the following.

1. If $\Gamma \vdash \varphi \dot{-} \widehat{2^{-k}}$ for every $k \in \mathbb{N}$, then $\varphi \in \Gamma$.

Classical completeness results for continuous logic

We say that Γ is *maximally consistent* if Γ is consistent and for every pair of L^+-wffs φ and ψ, we have the following.

1. If $\Gamma \vdash \varphi \dot{-} \widehat{2^{-k}}$ for every $k \in \mathbb{N}$, then $\varphi \in \Gamma$.
2. Either $\varphi \dot{-} \psi \in \Gamma$ or $\psi \dot{-} \varphi \in \Gamma$.

Classical completeness results for continuous logic

Let L be a signature and Γ a maximally consistent, Henkin complete L^+-theory.

Classical completeness results for continuous logic

Let L be a signature and Γ a maximally consistent, Henkin complete L^+-theory. Define the *Henkin* L^+*-pre-structure over* Γ, denoted $\mathfrak{M}'_\Gamma$, as follows.

Classical completeness results for continuous logic

Let L be a signature and Γ a maximally consistent, Henkin complete L^+-theory. Define the *Henkin L^+-pre-structure over* Γ, denoted $\mathfrak{M}'_\Gamma$, as follows.

1. $|\mathfrak{M}'_\Gamma|$ is the set of all terms of L.

Classical completeness results for continuous logic

Let L be a signature and Γ a maximally consistent, Henkin complete L^+-theory. Define the *Henkin L^+-pre-structure over* Γ, denoted $\mathfrak{M}'_\Gamma$, as follows.

1. $|\mathfrak{M}'_\Gamma|$ is the set of all terms of L.
2. For every constant symbol c of L, $c^{\mathfrak{M}'_\Gamma} := c$.

Classical completeness results for continuous logic

Let L be a signature and Γ a maximally consistent, Henkin complete L^+-theory. Define the *Henkin L^+-pre-structure over* Γ, denoted $\mathfrak{M}'_\Gamma$, as follows.

1. $|\mathfrak{M}'_\Gamma|$ is the set of all terms of L.
2. For every constant symbol c of L, $c^{\mathfrak{M}'_\Gamma} := c$.
3. For every N-ary function symbol f of L, and $t_0, \ldots, t_{N-1} \in |\mathfrak{M}'_\Gamma|$,

$$f^{\mathfrak{M}'_\Gamma}\left(t_0, \ldots, t_{N-1}\right) := f(t_0, \ldots, t_{N-1}).$$

Classical completeness results for continuous logic

Let L be a signature and Γ a maximally consistent, Henkin complete L^+-theory. Define the *Henkin L^+-pre-structure over* Γ, denoted $\mathfrak{M}'_\Gamma$, as follows.

1. $|\mathfrak{M}'_\Gamma|$ is the set of all terms of L.
2. For every constant symbol c of L, $c^{\mathfrak{M}'_\Gamma} := c$.
3. For every N-ary function symbol f of L, and $t_0, \ldots, t_{N-1} \in |\mathfrak{M}'_\Gamma|$,

$$f^{\mathfrak{M}'_\Gamma}\left(t_0, \ldots, t_{N-1}\right) := f\left(t_0, \ldots, t_{N-1}\right).$$

4. For every N-ary predicate symbol R of L, and $t_0, \ldots, t_{N-1} \in |\mathfrak{M}'_\Gamma|$,

$$R^{\mathfrak{M}'_\Gamma}\left(t_0, \ldots, t_{N-1}\right) := \sup\left\{\breve{p} \ : p \in \mathrm{Dyad}_L \text{ and } p \dot{-} R(t_0, \ldots, t_{N-1}) \in \Gamma\right\}.$$

Classical completeness results for continuous logic

Now define the *Henkin L^+-structure over* Γ, denoted $\mathfrak{M}_\Gamma$, to be the structure induced by the metric completion of $\mathfrak{M}'_\Gamma$ and the elementary morphism given in Theorem 6.9 of (Ben Yaacov and Pedersen, 2010).

Now define the *Henkin L^+-structure over* Γ, denoted $\mathfrak{M}_\Gamma$, to be the structure induced by the metric completion of $\mathfrak{M}'_\Gamma$ and the elementary morphism given in Theorem 6.9 of (Ben Yaacov and Pedersen, 2010).

Given a Henkin structure $\mathfrak{M}_\Gamma$, define the *basic assignment* on $\mathfrak{M}_\Gamma$ as $\sigma(x) := x$ for every variable symbol x of L.

Classical completeness results for continuous logic

Now define the *Henkin* L^+*-structure over* Γ, denoted $\mathfrak{M}_\Gamma$, to be the structure induced by the metric completion of $\mathfrak{M}'_\Gamma$ and the elementary morphism given in Theorem 6.9 of (Ben Yaacov and Pedersen, 2010).

Given a Henkin structure $\mathfrak{M}_\Gamma$, define the *basic assignment* on $\mathfrak{M}_\Gamma$ as $\sigma(x) := x$ for every variable symbol x of L. By a slight abuse of notation, when $\mathfrak{M}_\Gamma$ is a Henkin L^+-structure, by $\mathfrak{M}_\Gamma \models \varphi$ we mean $\mathfrak{M}_\Gamma, \sigma \models \varphi$ where σ is the basic assignment.

Classical completeness results for continuous logic

The following is a simple corollary from the proof of Theorem 9.4 in (Ben Yaacov and Pedersen, 2010).

Proposition

Let L be a signature and Γ a maximally consistent, Henkin complete L^+-theory. Then $\mathfrak{M}_\Gamma \models \Gamma$.

Effective extensions of complete, decidable theories

For the remainder of the presentation, we'll assume that L is an effectively encoded signature.

Effective extensions of complete, decidable theories

For the remainder of the presentation, we'll assume that L is an effectively encoded signature.

An L-theory T is called *complete* if there is an L-structure $\mathfrak{M}$ and assignment σ on $\mathfrak{M}$ such that $T = \{\varphi : \mathfrak{M}, \sigma \models \varphi\}$.

Effective extensions of complete, decidable theories

For the remainder of the presentation, we'll assume that L is an effectively encoded signature.

An L-theory T is called *complete* if there is an L-structure $\mathfrak{M}$ and assignment σ on $\mathfrak{M}$ such that $T = \{\varphi : \mathfrak{M}, \sigma \models \varphi\}$.

A complete L-theory T is then called *decidable* if

$$\varphi_T^\circ := \sup\{\varphi^{\mathfrak{M},\sigma} : \mathfrak{M}, \sigma \models T\}.$$

is a computable map from the effective encoding of the L-wffs to the computable real numbers.

Effective extensions of complete, decidable theories

For the remainder of the presentation, we'll assume that L is an effectively encoded signature.

An L-theory T is called *complete* if there is an L-structure $\mathfrak{M}$ and assignment σ on $\mathfrak{M}$ such that $T = \{\varphi : \mathfrak{M}, \sigma \models \varphi\}$.

A complete L-theory T is then called *decidable* if

$$\varphi_T^\circ := \sup\{\varphi^{\mathfrak{M},\sigma} : \mathfrak{M}, \sigma \models T\}.$$

is a computable map from the effective encoding of the L-wffs to the computable real numbers. An *index* of a complete, decidable L-theory is an index of this computable map.

Effective extensions of complete, decidable theories

Lemma (Lemma 4.6, Calvert, 2011)

There is an effective procedure which extends L to its Henkin extended signature L^+.

Effective extensions of complete, decidable theories

Lemma (Lemma 4.6, Calvert, 2011)

There is an effective procedure which extends L to its Henkin extended signature L^+.

Lemma (Lemma 4.6, Calvert, 2011)

There is uniformly computable sequence of finite sets of L^+-wffs $\{A_s\}_{s\in\mathbb{N}}$ such that $\bigcup_{s\in\mathbb{N}} A_s$ is Henkin complete and consistent with respect to any L-theory.

Effective extensions of complete, decidable theories

Lemma

There is an effective procedure which uniformly, given an index of a complete, decidable L-theory T, computes a sequence of finite sets of L^+-wffs $\{B_s\}_{s\in\mathbb{N}}$ such that for every pair of L^+-wffs φ and ψ, either φ and ψ are provably equivalent with respect to T, or exactly one of $\varphi \dot{-} \psi$ or $\psi \dot{-} \varphi$ is in $\bigcup_{s\in\mathbb{N}} B_s$.

Effective extensions of complete, decidable theories

Lemma

There is an effective procedure which uniformly, given an index of a complete, decidable L-theory T, computes a sequence of finite sets of L^+-wffs $\{B_s\}_{s\in\mathbb{N}}$ such that for every pair of L^+-wffs φ and ψ, either φ and ψ are provably equivalent with respect to T, or exactly one of $\varphi \dot{-} \psi$ or $\psi \dot{-} \varphi$ is in $\bigcup_{s\in\mathbb{N}} B_s$.

Proof sketch. At stage s, consider the first 2^s-many L^+-wffs of the form $\varphi \dot{-} \psi$ in the effective enumeration of the L^+-wffs.

Effective extensions of complete, decidable theories

Lemma

There is an effective procedure which uniformly, given an index of a complete, decidable L-theory T, computes a sequence of finite sets of L^+-wffs $\{B_s\}_{s\in\mathbb{N}}$ such that for every pair of L^+-wffs φ and ψ, either φ and ψ are provably equivalent with respect to T, or exactly one of $\varphi \dot{-} \psi$ or $\psi \dot{-} \varphi$ is in $\bigcup_{s\in\mathbb{N}} B_s$.

Proof sketch. At stage s, consider the first 2^s-many L^+-wffs of the form $\varphi \dot{-} \psi$ in the effective enumeration of the L^+-wffs.

For each of them, use the index of T to check if

$$\Big(\bigvee_{\theta\in B_{s-1}} \theta \vee (\varphi \dot{-} \psi)\Big)^{\circ}_{T} > 2^{-s}.$$

Effective extensions of complete, decidable theories

Lemma

There is an effective procedure which uniformly, given an index of a complete, decidable L-theory T, computes a sequence of finite sets of L^+-wffs $\{B_s\}_{s\in\mathbb{N}}$ such that for every pair of L^+-wffs φ and ψ, either φ and ψ are provably equivalent with respect to T, or exactly one of $\varphi \dot{-} \psi$ or $\psi \dot{-} \varphi$ is in $\bigcup_{s\in\mathbb{N}} B_s$.

Proof sketch. At stage s, consider the first 2^s-many L^+-wffs of the form $\varphi \dot{-} \psi$ in the effective enumeration of the L^+-wffs.

For each of them, use the index of T to check if

$$\Big(\bigvee_{\theta\in B_{s-1}} \theta \vee (\varphi \dot{-} \psi)\Big)^\circ_T > 2^{-s}.$$

If it is, add $\psi \dot{-} \varphi$. Otherwise, do nothing. ■

Effective extensions of complete, decidable theories

Lemma

There is an effective procedure which uniformly, given an index of a complete, decidable L-theory T, computes a sequence of finite sets of L^+-wffs $\{\Gamma_s\}_{s\in\mathbb{N}}$ such that $\bigcup_{S\in\mathbb{N}}\bigcap_{s\geqslant S}\Gamma_s$ is a maximally consistent, Henkin complete L^+-theory which extends T.

Effective extensions of complete, decidable theories

Lemma

There is an effective procedure which uniformly, given an index of a complete, decidable L-theory T, computes a sequence of finite sets of L^+-wffs $\{\Gamma_s\}_{s\in\mathbb{N}}$ such that $\bigcup_{S\in\mathbb{N}}\bigcap_{s\geqslant S}\Gamma_s$ is a maximally consistent, Henkin complete L^+-theory which extends T.

Proof sketch. Use the sets constructed in the previous lemmas, carefully put them together in the "right" way, and close them off by throwing out more and more refined inconsistencies to achieve a maximally consistent, Henkin complete set in the limit. ■

An effective completeness theorem

Theorem

There is an effective procedure which uniformly, given an index of a complete, decidable L-theory T, produces an index of a computable presentation of an L^+-structure $\mathfrak{M}$ such that $\mathfrak{M} \models T$.

An effective completeness theorem

Proof sketch. Compute L^+ as in Lemma 3 and, using the given index of a complete, decidable L-theory T, compute $\{\Gamma_s\}_{s\in\mathbb{N}}$ as in Lemma 6.

An effective completeness theorem

Proof sketch. Compute L^+ as in Lemma 3 and, using the given index of a complete, decidable L-theory T, compute $\{\Gamma_s\}_{s\in\mathbb{N}}$ as in Lemma 6.

Define $\Gamma := \bigcup_{S\in\mathbb{N}} \bigcap_{s\geqslant S} \Gamma_s$.

An effective completeness theorem

Proof sketch. Compute L^+ as in Lemma 3 and, using the given index of a complete, decidable L-theory T, compute $\{\Gamma_s\}_{s\in\mathbb{N}}$ as in Lemma 6.

Define $\Gamma := \bigcup_{S\in\mathbb{N}} \bigcap_{s\geqslant S} \Gamma_s$.

By Proposition 2, $\mathfrak{M}_\Gamma \models T$.

An effective completeness theorem

Since L^+ is effectively numbered, the set of constants of L^+ is also effectively numbered; let g be this effective numbering.

An effective completeness theorem

Since L^+ is effectively numbered, the set of constants of L^+ is also effectively numbered; let g be this effective numbering.

By construction, the algebra generated by $\mathrm{ran}(g)$ in $\mathfrak{M}_\Gamma$ is the set of all terms of L^+, that is, $|\mathfrak{M}'_\Gamma|$.

An effective completeness theorem

Since L^+ is effectively numbered, the set of constants of L^+ is also effectively numbered; let g be this effective numbering.

By construction, the algebra generated by $\mathrm{ran}(g)$ in $\mathfrak{M}_\Gamma$ is the set of all terms of L^+, that is, $|\mathfrak{M}'_\Gamma|$.

It follows that this algebra is dense in $|\mathfrak{M}_\Gamma|$, since $|\mathfrak{M}_\Gamma|$ is the metric completion of $|\mathfrak{M}'_\Gamma|$.

An effective completeness theorem

Since L^+ is effectively numbered, the set of constants of L^+ is also effectively numbered; let g be this effective numbering.

By construction, the algebra generated by $\mathrm{ran}(g)$ in $\mathfrak{M}_\Gamma$ is the set of all terms of L^+, that is, $|\mathfrak{M}'_\Gamma|$.

It follows that this algebra is dense in $|\mathfrak{M}_\Gamma|$, since $|\mathfrak{M}_\Gamma|$ is the metric completion of $|\mathfrak{M}'_\Gamma|$. Thus $(\mathfrak{M}_\Gamma, g)$ is a presentation of $\mathfrak{M}_\Gamma$.

An effective completeness theorem

Since L^+ is effectively numbered, the set of constants of L^+ is also effectively numbered; let g be this effective numbering.

By construction, the algebra generated by $\mathrm{ran}(g)$ in $\mathfrak{M}_\Gamma$ is the set of all terms of L^+, that is, $|\mathfrak{M}'_\Gamma|$.

It follows that this algebra is dense in $|\mathfrak{M}_\Gamma|$, since $|\mathfrak{M}_\Gamma|$ is the metric completion of $|\mathfrak{M}'_\Gamma|$. Thus $\left(\mathfrak{M}_\Gamma, g\right)$ is a presentation of $\mathfrak{M}_\Gamma$.

Moreover, by querying the Γ_s sets in the "right" way, one can compute an approximation of the value of any predicate on any rational points.

An effective completeness theorem

Since L^+ is effectively numbered, the set of constants of L^+ is also effectively numbered; let g be this effective numbering.

By construction, the algebra generated by $\mathrm{ran}(g)$ in $\mathfrak{M}_\Gamma$ is the set of all terms of L^+, that is, $|\mathfrak{M}'_\Gamma|$.

It follows that this algebra is dense in $|\mathfrak{M}_\Gamma|$, since $|\mathfrak{M}_\Gamma|$ is the metric completion of $|\mathfrak{M}'_\Gamma|$. Thus $(\mathfrak{M}_\Gamma, g)$ is a presentation of $\mathfrak{M}_\Gamma$.

Moreover, by querying the Γ_s sets in the "right" way, one can compute an approximation of the value of any predicate on any rational points. Therefore, $(\mathfrak{M}_\Gamma, g)$ is a computable presentation. ■

QUESTIONS?

Works Cited

Itai Ben Yaacov, Alexander Berenstein, C. Ward Henson, and Alexander Usvyatsov, *Model theory for metric structures*, **Model Theory with Applications to Algebra and Analysis, vol. 2**, (Zoé Chatzidakis, Dugald Macpherson, Anand Pillay, and Alex Wilkie, editors) London Math Society Lecture Note Series, vol. 350, London, 2008, pp. 315–427.

Itai Ben Yaacov and Arthur Paul Pedersen, *A proof of completeness for continuous first-order logic*, **The Journal of Symbolic Logic**, vol. 75 (2010), pp. 168–190.

Itai Ben Yaacov and Alexander Usvyatsov, *Continuous first order logic and local stability*, **Transactions of the American Mathematical Society**, vol. 362, no. 10 (2010), pp. 5213–5259.

Works Cited

Johanna N.Y. Franklin and Timothy H. McNicholl, *Degrees of and lowness for isometric isomorphism*, **Journal of Logic & Analysis**, vol. 12, no. 6 (2020), pp. 1–23.

Wesley Calvert, *Metric structures and probabilistic computation*, **Theoretical Computer Science**, vol. 412 (2011), pp. 2766–2775.

V. S. Harizanov, *Pure computable model theory*, **Handbook of Recursive Mathematics, vol. 1**, (Yu. L. Ershov, S. S. Goncharov, A. Nerode, J.B. Remmel, and V. W. Marek, editors) Studies in Logic and the Foundations of Mathematics, 138-139, North-Holland, Amsterdam, 1998, pp. 3–114.