Department of Mathematics, University of Udine

Cantor-Bendixson theorem in the Weihrauch lattice

Vittorio Cipriani
joint work with Alberto Marcone and Manlio Valenti

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The largest perfect subset of X is called the *perfect kernel* of X (denoted by $\mathsf{PK}(X)$).
$X \setminus \mathsf{PK}(X)$ is called the *scattered part* of X.

Theorems as the one above can be written as:

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Theorems as problems

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given in **input** x s.t. $\varphi(x)$, produce as **output** a y s.t. $\psi(x, y)$

N.B. we will show that, for many theorems, there may be many "natural" ways to phrase them as a computational problem.

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Now we can think of a computational problem as a (possibly partial) *multivalued functions* $f :\subseteq X \rightrightarrows Y$, where X, Y are represented spaces.

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Weihrauch Reducibility

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- Given a name p for $x \in dom(f)$, $\Phi(p)$ is a name for $z \in dom(g)$;
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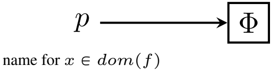

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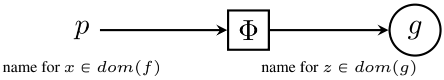

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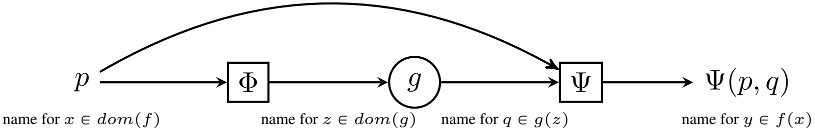

In rev. math. we have the so called "big-five phenomenon", that, informally, says that (most of) theorems in classical mathematics are equivalent to one of these five subsystems of SOA:

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- WKL_0
- ACA_0
- ATR_0
- Π^1_1-CA_0

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- $\mathsf{ACA}_0 \rightsquigarrow$ (iterations of) lim
- ATR_0
- $\Pi^1_1\text{-}\mathsf{CA}_0$

Which are the representatives (in the Weihrauch lattice) of the big five? Most of the work so far is about the first three.

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ATR_0 and Π^1_1-CA_0

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For Π^1_1-CA_0 the situation is quite clear. We have a natural candidate that is Π^1_1-$\mathsf{CA} \equiv_W \widehat{\chi_{\Pi^1_1}}$, where $\chi_{\Pi^1_1}$ is the characteristic function of the collection of well-founded trees.

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> The paralleization of a multi-valued function f, denoted by $\widehat{f}$, takes in input a countable sequence of instances of f and solves all of them.

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$$\mathsf{C}_{\mathbb{N}^\mathbb{N}} : \quad \textbf{Input} \text{ an ill-founded tree } T \subseteq \mathbb{N}^{<\mathbb{N}} \quad \textbf{Output} \text{ a path through } T.$$

$\mathsf{UC}_{\mathbb{N}^\mathbb{N}}$ is the restriction of $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$ to trees with a unique path.

Theorem

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List: **Input** a tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ without perfect subtrees
Output a list $(p_i)_{i\in\omega}$ of all the paths in $[T]$ and $n = |[T]|$.

Lemma (Kihara, Marcone, Pauly)

$\mathsf{wList} \equiv_W \mathsf{List} \equiv_W \mathsf{UC}_{\mathbb{N}^\mathbb{N}} <_W \mathsf{PTT}_1 \equiv_W \mathsf{C}_{\mathbb{N}^\mathbb{N}}$.

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Lemma (C., Marcone, Valenti)

$\mathsf{UC}_{\mathbb{N}^\mathbb{N}} <_W \mathsf{PST}_{1,\mathbb{N}^\mathbb{N}} <_W \mathsf{C}_{\mathbb{N}^\mathbb{N}}$.

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Lemma

$\mathsf{wList}_{2^\mathbb{N}} <_W \mathsf{List}_{2^\mathbb{N}} <_W \mathsf{UC}_{\mathbb{N}^\mathbb{N}}$.

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For the *perfect kernel*...

PK_X: **Input** a closed subset A of X
Output the perfect kernel of A.

$\mathsf{PK}_{\mathsf{Tr}}$: **Input** a tree T
Output the largest perfect subtree of T.

For the *scattered part*...

$\mathsf{wScList}_X$: **Input** a closed subset A of X
Output a "weak" list of the scattered part of A.

ScList_X: **Input** a closed set A of X
Output a "strong" list of the scattered part of A.

For the *scattered part*...

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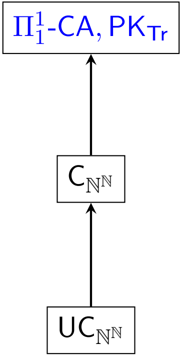
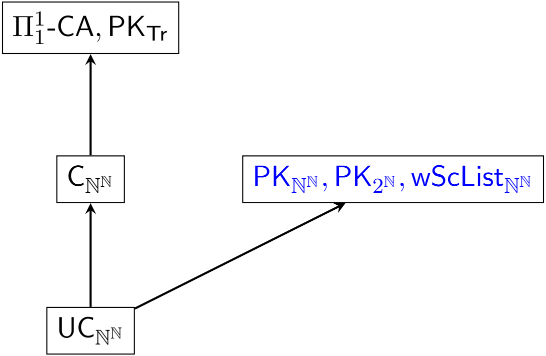
As before, the "listing principles" have different strength in $\mathbb{N}^{\mathbb{N}}$ and $2^{\mathbb{N}}$, while the "perfect kernel principles" haven't.



The equivalence is due to **Hirst**.



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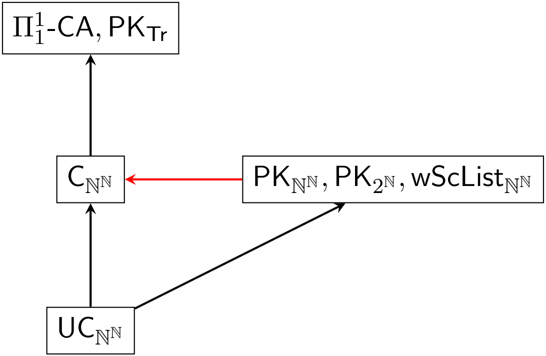
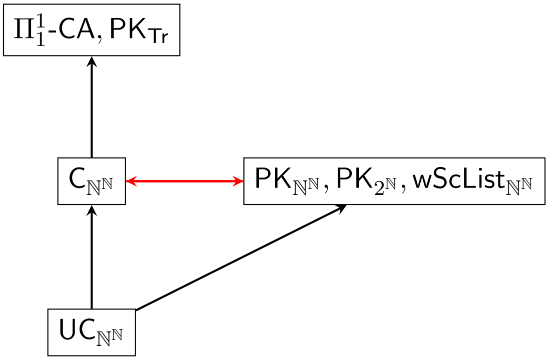


C., Marcone, Valenti

Proof.

$\chi_{\Pi^1_1} \leqslant_W \mathsf{LPO} * \mathsf{PK}_{\mathbb{N}^\mathbb{N}}$, and since $\mathsf{PK}_{\mathbb{N}^\mathbb{N}}$ is parallelizable Π^1_1-$\mathsf{CA} \leqslant_W \widehat{\mathsf{LPO}} * \mathsf{PK}_{\mathbb{N}^\mathbb{N}}$. On the other hand, $\mathsf{C}_{\mathbb{N}^\mathbb{N}} \equiv_W \widehat{\mathsf{LPO}} * \mathsf{C}_{\mathbb{N}^\mathbb{N}}$. □

N.B.: Both $\mathsf{PK}_{2^\mathbb{N}}$ and $\mathsf{wScList}_{\mathbb{N}^\mathbb{N}}$ can be used to turn a Π^1_1 question into a Σ^0_1 one.



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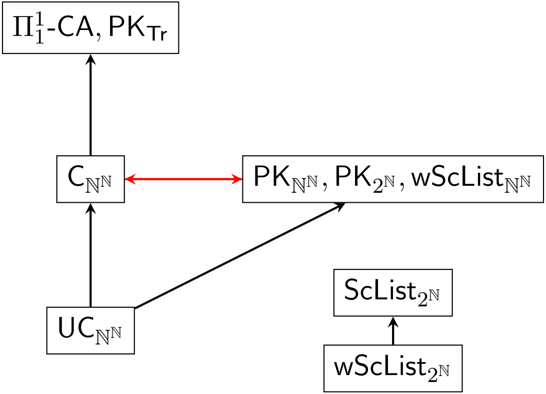
The proof uses a concept introduced by (Dzhafarov, Solomon, Yokoyama), i.e. the *first-order part* of a multi-valued function f. Informally, it is the hardest multi-valued function with codomain $\mathbb{N}$ computed by f.

Proof.

${}^1\mathsf{C}_{\mathbb{N}^\mathbb{N}} \equiv_W \Sigma^1_1\text{-}\mathsf{C}_\mathbb{N}$, ${}^1\mathsf{PK}_{\mathbb{N}^\mathbb{N}} \equiv_W \Delta^1_1\text{-}\mathsf{C}_\mathbb{N}$ and $\Delta^1_1\text{-}\mathsf{C}_\mathbb{N} <_W \Sigma^1_1\text{-}\mathsf{C}_\mathbb{N}$. □



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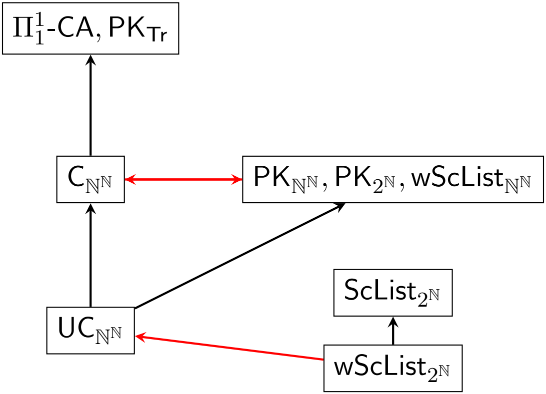
Again, first-order part.



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Proof.

$\chi_{\Pi^1_1} \leqslant_W \mathsf{LPO}' * \mathsf{wScList}_{2^\mathbb{N}}$. □

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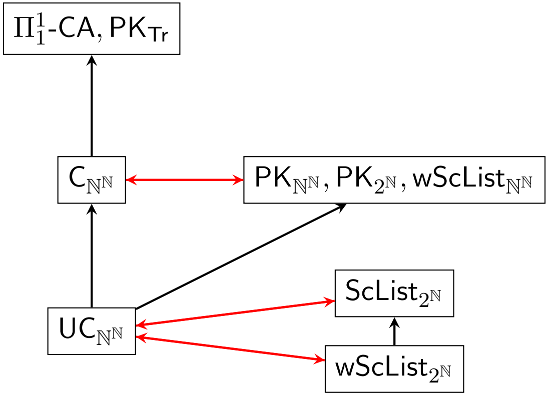

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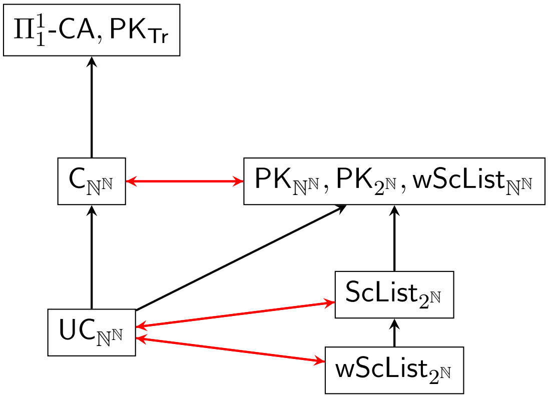

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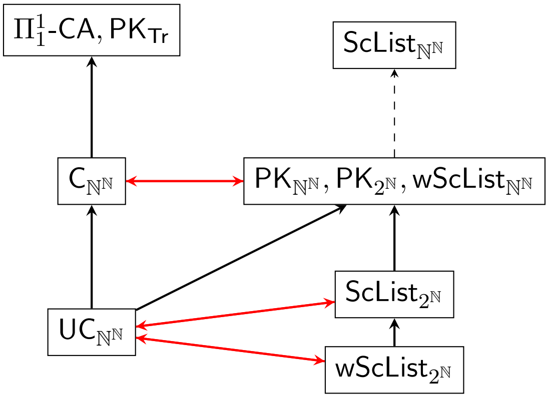
Proof.

$\mathsf{PK}_{\mathbb{N}^\mathbb{N}}$ is parallelizable and computes both $\mathsf{wScList}_{2^\mathbb{N}}$ and the cardinality of the scattered part. With these two elements we can compute a "strong" list. □


Π^1_1-CA, $\mathsf{PK}_{\mathrm{Tr}}$
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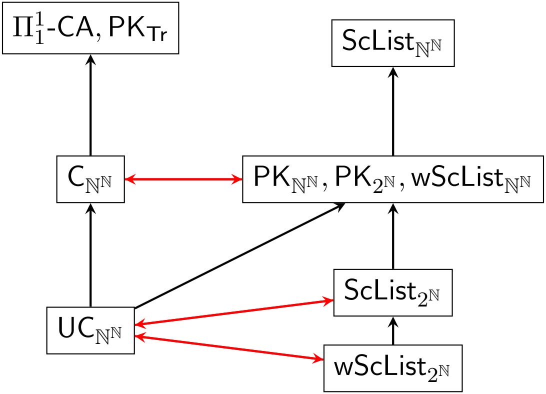

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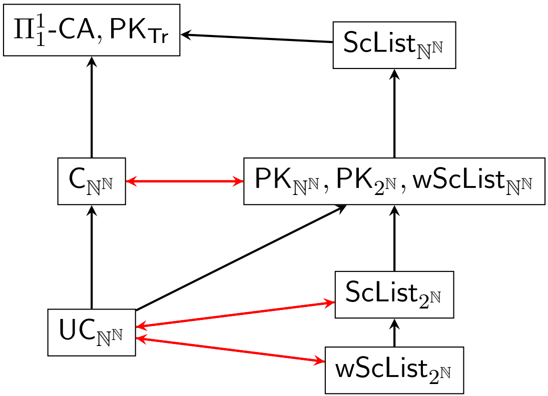
Proof.

$\chi_{\Pi^1_1} \leqslant_W {}^1\mathsf{ScList}_{\mathbb{N}^\mathbb{N}}$ while $\chi_{\Pi^1_1} \not\leqslant_W {}^1\mathsf{PK}_{\mathbb{N}^\mathbb{N}} \equiv_W \Delta^1_1\text{-}\mathsf{C}_{\mathbb{N}}$. □



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Proof.

Π^1_1-CA is strong enough to compute both $\mathsf{wScList}_{\mathbb{N}^\mathbb{N}}$ and the cardinality of the scattered part. With these two elements we can compute a "strong" list. Strictness comes in a few slides. □

The "full" Cantor-Bendixson (trees)

$\mathsf{wCB}_{\mathsf{Tr}}$: **Input** a tree T,
Output $\mathsf{PK}_{\mathsf{Tr}}(T)$ and $\mathsf{wScList}_{\mathbb{N}^\mathbb{N}}(T)$.

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$\mathsf{wCB_{Tr}}$: **Input** a tree T,
Output $\mathsf{PK_{Tr}}(T)$ and $\mathsf{wScList}_{\mathbb{N}^{\mathbb{N}}}(T)$.

$\mathsf{CB_{Tr}}$: **Input** a tree T
Output $\mathsf{PK_{Tr}}(T)$ and $\mathsf{ScList}_{\mathbb{N}^{\mathbb{N}}}(T)$.

From the previous reductions and from the fact that Π^1_1-CA is closed under $\times$:

The "full" Cantor-Bendixson (trees)

$\mathsf{wCB_{Tr}}$: **Input** a tree T,
Output $\mathsf{PK_{Tr}}(T)$ and $\mathsf{wScList}_{\mathbb{N}^\mathbb{N}}(T)$.

$\mathsf{CB_{Tr}}$: **Input** a tree T
Output $\mathsf{PK_{Tr}}(T)$ and $\mathsf{ScList}_{\mathbb{N}^\mathbb{N}}(T)$.

From the previous reductions and from the fact that Π^1_1-CA is closed under $\times$:
Π^1_1-$\mathsf{CA} \equiv_W \mathsf{PK_{Tr}} \times \mathsf{wScList}_{\mathbb{N}^\mathbb{N}} \equiv_W \mathsf{PK_{Tr}} \times \mathsf{ScList}_{\mathbb{N}^\mathbb{N}}$

$\mathsf{wCB}_{\mathsf{Tr}}$: **Input** a tree T,
Output $\mathsf{PK}_{\mathsf{Tr}}(T)$ and $\mathsf{wScList}_{\mathbb{N}^{\mathbb{N}}}(T)$.

$\mathsf{CB}_{\mathsf{Tr}}$: **Input** a tree T
Output $\mathsf{PK}_{\mathsf{Tr}}(T)$ and $\mathsf{ScList}_{\mathbb{N}^{\mathbb{N}}}(T)$.

From the previous reductions and from the fact that Π^1_1-CA is closed under $\times$:
$\Pi^1_1\text{-}\mathsf{CA} \equiv_{\mathrm{W}} \mathsf{PK}_{\mathsf{Tr}} \times \mathsf{wScList}_{\mathbb{N}^{\mathbb{N}}} \equiv_{\mathrm{W}} \mathsf{PK}_{\mathsf{Tr}} \times \mathsf{ScList}_{\mathbb{N}^{\mathbb{N}}}$
$\rightsquigarrow \Pi^1_1\text{-}\mathsf{CA} \equiv_{\mathrm{W}} \mathsf{wCB}_{\mathsf{Tr}} \equiv_{\mathrm{W}} \mathsf{CB}_{\mathsf{Tr}}$

wCB_X: **Input** a closed set A of X,
Output $\mathsf{PK}_X(A)$ and $\mathsf{wScList}_X(A)$.

The "full" Cantor-Bendixson (closed sets)

wCB_X: **Input** a closed set A of X,
Output $\mathsf{PK}_X(A)$ and $\mathsf{wScList}_X(A)$.

CB_X: **Input** a closed set A of X
Output $\mathsf{PK}_X(A)$ and $\mathsf{ScList}_X(A)$.

wCB_X: **Input** a closed set A of X,
Output $\mathsf{PK}_X(A)$ and $\mathsf{wScList}_X(A)$.

CB_X: **Input** a closed set A of X
Output $\mathsf{PK}_X(A)$ and $\mathsf{ScList}_X(A)$.

From the previous reductions and from the fact that $\mathsf{PK}_{\mathbb{N}^\mathbb{N}}$ is closed under $\times$:
$\mathsf{PK}_{\mathbb{N}^\mathbb{N}} \equiv_W \mathsf{PK}_{2^\mathbb{N}} \times \mathsf{wScList}_{2^\mathbb{N}} \equiv_W \mathsf{PK}_{2^\mathbb{N}} \times \mathsf{ScList}_{2^\mathbb{N}}$

wCB_X: **Input** a closed set A of X,
Output $\mathsf{PK}_X(A)$ and $\mathsf{wScList}_X(A)$.

CB_X: **Input** a closed set A of X
Output $\mathsf{PK}_X(A)$ and $\mathsf{ScList}_X(A)$.

From the previous reductions and from the fact that $\mathsf{PK}_{\mathbb{N}^\mathbb{N}}$ is closed under $\times$:

$\mathsf{PK}_{\mathbb{N}^\mathbb{N}} \equiv_W \mathsf{PK}_{2^\mathbb{N}} \times \mathsf{wScList}_{2^\mathbb{N}} \equiv_W \mathsf{PK}_{2^\mathbb{N}} \times \mathsf{ScList}_{2^\mathbb{N}}$

$\rightsquigarrow \mathsf{PK}_{\mathbb{N}^\mathbb{N}} \equiv_W \mathsf{wCB}_{2^\mathbb{N}} \equiv_W \mathsf{CB}_{2^\mathbb{N}}$

wCB_X: **Input** a closed set A of X,
Output $\mathsf{PK}_X(A)$ and $\mathsf{wScList}_X(A)$.

CB_X: **Input** a closed set A of X
Output $\mathsf{PK}_X(A)$ and $\mathsf{ScList}_X(A)$.

From the previous reductions and from the fact that $\mathsf{PK}_{\mathbb{N}^\mathbb{N}}$ is closed under $\times$:

$\mathsf{PK}_{\mathbb{N}^\mathbb{N}} \equiv_W \mathsf{PK}_{2^\mathbb{N}} \times \mathsf{wScList}_{2^\mathbb{N}} \equiv_W \mathsf{PK}_{2^\mathbb{N}} \times \mathsf{ScList}_{2^\mathbb{N}}$

$\rightsquigarrow \mathsf{PK}_{\mathbb{N}^\mathbb{N}} \equiv_W \mathsf{wCB}_{2^\mathbb{N}} \equiv_W \mathsf{CB}_{2^\mathbb{N}}$

$\mathsf{PK}_{\mathbb{N}^\mathbb{N}} \equiv_W \mathsf{wScList}_{\mathbb{N}^\mathbb{N}} \times \mathsf{PK}_{\mathbb{N}^\mathbb{N}}$

wCB_X: **Input** a closed set A of X,
Output $\mathsf{PK}_X(A)$ and $\mathsf{wScList}_X(A)$.

CB_X: **Input** a closed set A of X
Output $\mathsf{PK}_X(A)$ and $\mathsf{ScList}_X(A)$.

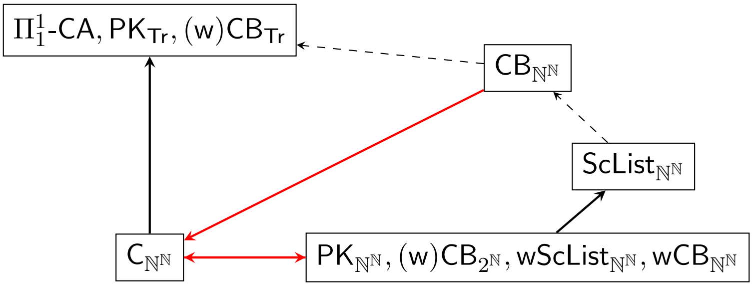
From the previous reductions and from the fact that $\mathsf{PK}_{\mathbb{N}^\mathbb{N}}$ is closed under $\times$:

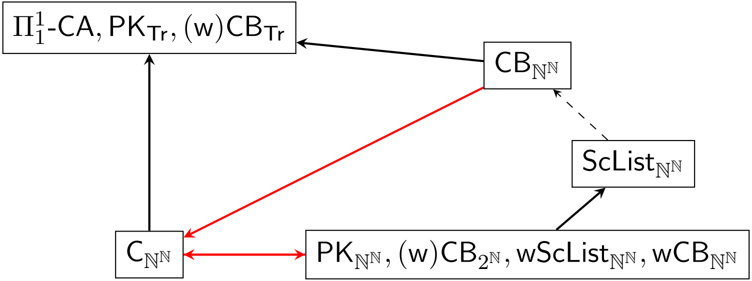
$\mathsf{PK}_{\mathbb{N}^\mathbb{N}} \equiv_W \mathsf{PK}_{2^\mathbb{N}} \times \mathsf{wScList}_{2^\mathbb{N}} \equiv_W \mathsf{PK}_{2^\mathbb{N}} \times \mathsf{ScList}_{2^\mathbb{N}}$

$\rightsquigarrow \mathsf{PK}_{\mathbb{N}^\mathbb{N}} \equiv_W \mathsf{wCB}_{2^\mathbb{N}} \equiv_W \mathsf{CB}_{2^\mathbb{N}}$

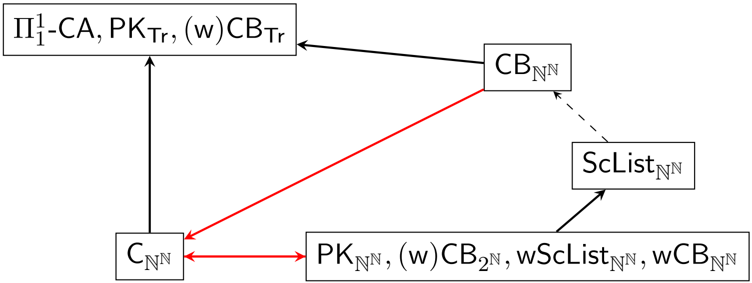
$\mathsf{PK}_{\mathbb{N}^\mathbb{N}} \equiv_W \mathsf{wScList}_{\mathbb{N}^\mathbb{N}} \times \mathsf{PK}_{\mathbb{N}^\mathbb{N}}$

$\rightsquigarrow \mathsf{PK}_{\mathbb{N}^\mathbb{N}} \equiv_W \mathsf{wCB}_{\mathbb{N}^\mathbb{N}}$


Π^1_1-CA, $\mathsf{PK}_{\mathrm{Tr}}$, $\mathsf{(w)CB}_{\mathrm{Tr}}$
$\mathsf{CB}_{\mathbb{N}^\mathbb{N}}$
$\mathsf{ScList}_{\mathbb{N}^\mathbb{N}}$
$\mathsf{C}_{\mathbb{N}^\mathbb{N}}$
$\mathsf{PK}_{\mathbb{N}^\mathbb{N}}$, $\mathsf{(w)CB}_{2^\mathbb{N}}$, $\mathsf{wScList}_{\mathbb{N}^\mathbb{N}}$, $\mathsf{wCB}_{\mathbb{N}^\mathbb{N}}$

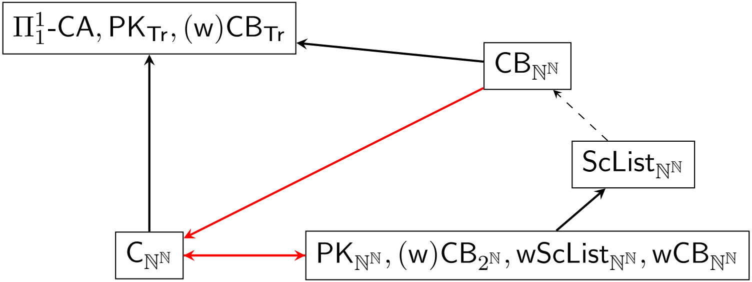


Strictness follows from the fact that if f is single-valued and $f \leqslant_W \mathsf{CB}_{\mathbb{N}^\mathbb{N}}$, then f always have an HYP solution (relative to the input).



Open Questions:

- $\mathsf{C}_{\mathbb{N}^\mathbb{N}} \leqslant_\mathrm{W} \mathsf{CB}_{\mathbb{N}^\mathbb{N}}$?
- $\mathsf{ScList}_{\mathbb{N}^\mathbb{N}} <_\mathrm{W} \mathsf{CB}_{\mathbb{N}^\mathbb{N}}$?
- What about PK_X and (w)CB_X for other Polish spaces X?



Open Questions:

- $\mathsf{C}_{\mathbb{N}^\mathbb{N}} \leqslant_W \mathsf{CB}_{\mathbb{N}^\mathbb{N}}$?
- $\mathsf{ScList}_{\mathbb{N}^\mathbb{N}} <_W \mathsf{CB}_{\mathbb{N}^\mathbb{N}}$?
- What about PK_X and $(\mathsf{w})\mathsf{CB}_X$ for other Polish spaces X?

Thanks for your attention!