AN ALGORITHM FOR REORIENTING INFINITE PSEUDO-TRANSITIVE GRAPHS

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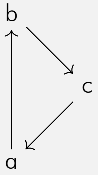
Theorem (Ghouila-Houri)

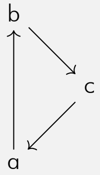
For each pseudo-transitive oriented graph there exists a transitive reorientation

A graph $(V, \rightarrow)$ is <u>pseudo-transitive</u> if for each $a, b, c \in V$
if $a \rightarrow b \rightarrow c$ then either $a \rightarrow c$ or $c \rightarrow a$.

$(V, <)$ is a <u>reorientation</u> of $(V, \rightarrow)$ if
$a, b \in V$ are $\rightarrow$-comparable if and only if they are $<$-comparable.



pseudo-transitive graph



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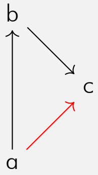

transitive reorientation

Theorem (Ghouila-Houri, Gilmore Hoffman)

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Theorem (Hirst)

The following statement is NOT computably true:
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INPUT a pseudo-transitive oriented graph $(V, \to)$ (possibly infinite)

and gives as

OUTPUT a transitive oriented graph $(V, \prec)$ such that $a \mid_{\to} b \Leftrightarrow a \mid_{\prec} b$

Theorem

- there exists an on-line (incremental) algorithm to transitively reorient pseudo-transitive oriented graphs;

Let $(\{v_n \mid n \in \mathbb{N}\}, \to)$ be a pseudo-transitive oriented graph

INPUT at stage n the algorithm reads

- the n^{th} vertex v_n of $(V, \to)$,
- the edges connecting v_n to $\{v_0, \dots, v_{n-1}\}$

At most n new bits of information at each step

OUTPUT at stage n the algorithm outputs

- a transitive reorientation of all the edges connecting v_n to $\{v_0, \dots, v_{n-1}\}$,
- preserves the reorientations already set at previous stages

No mind changes

Theorem

- there exists an on-line (incremental) algorithm to transitively reorient pseudo-transitive oriented graphs;
- RCA_0 proves that every countable pseudo-transitive oriented graph has a transitive reorientation;
- every computable pseudo-transitive oriented graph has a computable transitive reorientation;
- the multi-valued function that maps a countable pseudo-transitive oriented graph to the set of its transitive reorientations is computable;
- Player II has a winning strategy for the following game: starting from the empty graph, at step $s+1$ player I plays a pseudo-transitive extension $(V_s \cup \{x_s\}, \to_{s+1})$ of the pseudo-transitive oriented graph $(V_s, \to_s)$ he played at step s. Player II replies with a transitive reorientation $\prec_{s+1}$ of $\to_{s+1}$ such that $\prec_{s+1}$ extends $\prec_s$ she defined at step s. Player II wins if and only if she is always able to play.

There exists an on-line algorithm which takes as

INPUT a pseudo-transitive oriented graph $(V, \to)$ (possibly infinite)

and gives as

OUTPUT a transitive oriented graph $(V, \prec)$ such that $a \,|_{\to}\, b \Leftrightarrow a \,|_{\prec}\, b$

Obstacles: transitive triangle

$a \prec \qquad b$

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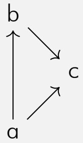
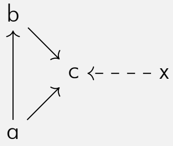


$a \prec c \prec b$

Obstacles: transitive triangle



$a \prec c \prec b$ is NOT extendible

> $(V, \to, \prec)$ is extendible if for every $(V \cup \{x\}, \to')$, pseudo-transitive extension of $(V, \to)$, there exists $\prec' \supseteq \prec$ transitive reorientation of $(V \cup \{x\}, \to')$

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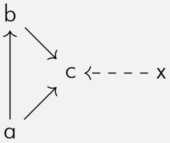

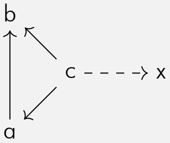

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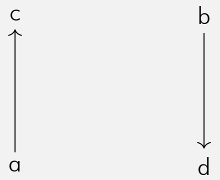
Obstacles: $2 \oplus 2$



$$a \prec c, \quad d \prec b$$

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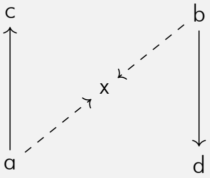

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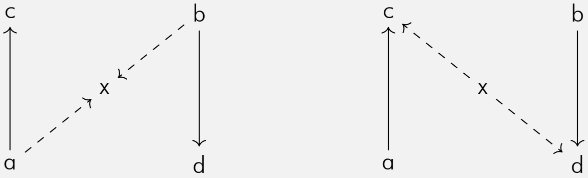


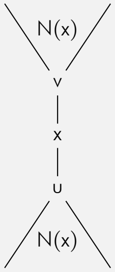
a $\prec$ c, d $\prec$ b is NOT extendible

- we isolate the two possibly “problematic” situations
- we state two properties which must be preserved in order for a reorientation to be extendible
- we design an algorithm which preserves the properties, i.e. that the algorithm is correct and complete

Let $(V, \rightarrow)$ be a pseudo-transitive graph and $\prec$ a transitive reorientation.

If $(V \cup \{x\}, \rightarrow')$ is a pseudo-transitive extension of $(V, \rightarrow)$ let

$$
\begin{aligned}
N(x) &= \{a \in V \mid a \rightarrow' x \vee x \rightarrow' a\} \\
N^+(x) &= \{v \in N(x) \mid \forall w\,(v \prec w \Rightarrow w \in N(x))\}; \\
N^-(x) &= \{u \in N(x) \mid \forall w\,(w \prec u \Rightarrow w \in N(x))\}.
\end{aligned}
$$



Let $(V, \to)$ be a pseudo-transitive graph and $\prec$ a transitive reorientation.

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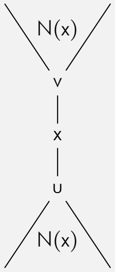


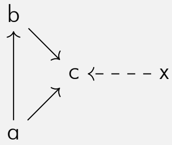
If $(V, \to, \prec)$ is extendible,
then for any $(V \cup \{x\}, \to')$ pseudo-transitive extension of $(V, \to)$ we have:

- $N(x) = N^+(x) \cup N^-(x)$;
- $N^-(x) \setminus N^+(x) \prec N^+(x) \setminus N^-(x)$.

Back to transitive triangles

Let $(V, \to)$ be a pseudo-transitive graph and $\prec$ a transitive reorientation.

- For any $(V \cup \{x\}, \to')$ pseudo-transitive extension of $(V, \to)$ the following are equivalent:
  - (i) $N(x) = N^+(x) \cup N^-(x)$;
  - (ii) $\forall a, b, c \in V \, (a \prec c \prec b \wedge x -' c \Rightarrow x -' a \vee x -' b)$.



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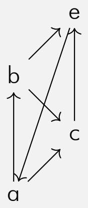
- Φ and Ψ are satisfied



> $\varphi(a, b, c) :=$ there exists $e_0, \ldots, e_n \in V$ such that:
> (φ_1) $c \to e_0$;
> (φ_2) $\forall i < n\, ((a \to e_i \wedge b \to e_i \to e_{i+1}) \vee (e_{i+1} \to e_i \to b \wedge e_i \to a))$;
> (φ_3) $a \to e_n \to b \vee b \to e_n \to a$.
> Then Φ is $\forall a, b, c \in V\, (a \to c \leftarrow b \wedge a \prec c \prec b \Rightarrow \varphi(a, b, c))$

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Let $(V, \rightarrow)$ be a pseudo-transitive graph and $\prec$ a transitive reorientation.
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- For any $(V \cup \{x\}, \rightarrow')$ pseudo-transitive extension of $(V, \rightarrow)$ the following are equivalent:
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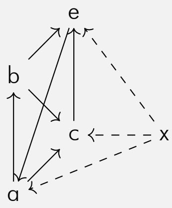

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> (φ_3) $a \rightarrow e_n \rightarrow b \vee b \rightarrow e_n \rightarrow a$.
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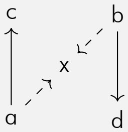
Back to $2 \oplus 2$

Let $(V, \to)$ be a pseudo-transitive graph and
$\prec$ a transitive reorientation which satisfy Φ and Ψ.

- For any $(V \cup \{x\}, \to')$ pseudo-transitive extension of $(V, \to)$ the following are equivalent:
  - (i) $N^-(x) \setminus N^+(x) \prec N^+(x) \setminus N^-(x)$;
  - (ii) $\forall a, b, c, d \in V\,(a \mid b \wedge c \mid d \wedge a \prec c \wedge d \prec b \wedge x -' a \wedge x -' b \Rightarrow x -' d \vee x -' c)$.



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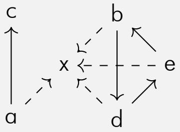
- Θ is satisfied



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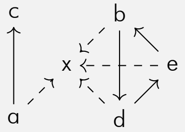

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- Θ is satisfied

  $\theta(a, b, c, d) :=$ there exists $e_0, \ldots, e_n \in V$ such that:

  (θ_1) $e_0 \to b$;

  (θ_2) $\forall i < n\, (e_{i+1} \to e_i \to d)$;

  (θ_3) $d \to e_n$;

  (θ_4) $e_n \mid a$.

  Then Θ is $\forall a, b, c, d \in V\, (a \to c \wedge b \to d \wedge a \mid b \wedge c \mid d \wedge a \prec c \wedge d \prec b \Rightarrow \theta(a, b, c, d) \vee \theta(b, a, d, c))$

Algorithm

Let $(V, \to)$ be a pseudo-transitive oriented graph and $\prec$ a reorientation.
If $(V \cup \{x\}, \to')$ is a pseudo-transitive extension of $\to$, we define inductively the following subsets of $N(x)$:

$$S_0^-(x) = N^-(x) \setminus N^+(x);$$

$$S_0^+(x) = N^+(x) \setminus N^-(x);$$

$$S_i(x) = S_i^-(x) \cup S_i^+(x);$$

$$S_{i+1}^-(x) = \{a \in N(x) \setminus \bigcup_{j \leq i} S_j(x) \mid \exists s \in S_i^-(x)(a \mid s)\};$$

$$S_{i+1}^+(x) = \{a \in N(x) \setminus \bigcup_{j \leq i} S_j(x) \mid \exists s \in S_i^+(x)(a \mid s)\}.$$

Let $S^+(x) = \bigcup_{i \in \mathbb{N}} S_i^+(x)$, $S^-(x) = \bigcup_{i \in \mathbb{N}} S_i^-(x)$.
Let $T(x) = N(x) \setminus S(x)$.

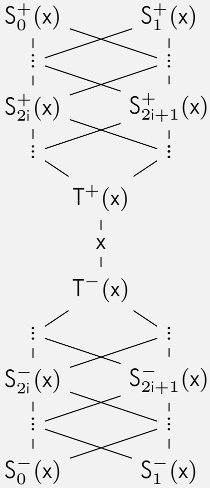
Algorithm

INPUT $(\{v_n \mid n \in \mathbb{N}\}, \rightarrow)$ pseudo-transitive oriented graph

PARTIAL OUTPUT $(\{v_0, \ldots, v_n\}, \prec)$ transitive reorientation

A STEP $(\{v_0, \ldots, v_n, x\}, \prec')$ such that for each $i \leq n$

- if $v_i \notin N(x)$ let $v_i \not\prec' x$ and $x \not\prec' v_i$,
- if $v_i \in S^-(x)$ let $v_i \prec' x$,
- if $v_i \in S^+(x)$ let $x \prec' v_i$,
- if $v_i \in T(x)$ then
  - (a) if there exists $j < i$ such that $v_i \prec v_j \prec' x$ let $v_i \prec' x$,
  - (b) if there exists $j < i$ such that $x \prec' v_j \prec v_i$ let $x \prec' v_i$,
  - (c) otherwise let $v_i \prec' x$ if $v_i \rightarrow' x$ and $x \prec' v_i$ if $x \rightarrow' v_i$.



A STEP $(\{v_0, \ldots, v_n, x\}, \prec')$ such that for each $i \leq n$

- if $v_i \notin N(x)$ let $v_i \not\prec' x$ and $x \not\prec' v$,
- if $v_i \in S^-(x)$ let $v_i \prec' x$,
- if $v_i \in S^+(x)$ let $x \prec' v_i$,
- if $v_i \in T(x)$ then
  (a) if there exists $j < i$ such that $v_i \prec v_j \prec' x$ let $v_i \prec' x$,
  (b) if there exists $j < i$ such that $x \prec' v_j \prec v_i$ let $x \prec' v_i$,
  (c) otherwise let $v_i \prec' x$ if $v_i \to' x$ and $x \prec' v_i$ if $x \to' v_i$.

We need to check that

- $\prec'$ is a reorientation
- $\prec'$ is transitive

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We need to check that

- $\prec'$ is a **reorientation**
- $\prec'$ is transitive

If $S^-(x) \cap S^+(x) = \emptyset$ and $S(x) \cap T(x) = \emptyset$, then $\prec'$ is a reorientation

A STEP $(\{v_0, \ldots, v_n, x\}, \prec')$ such that for each $i \leq n$

- if $v_i \notin N(x)$ let $v_i \not\prec' x$ and $x \not\prec' v$,
- if $v_i \in S^-(x)$ let $v_i \prec' x$,
- if $v_i \in S^+(x)$ let $x \prec' v_i$,
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  (a) if there exists $j < i$ such that $v_i \prec v_j \prec' x$ let $v_i \prec' x$,
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We need to check that

- $\prec'$ is a **reorientation**
- $\prec'$ is transitive

If $(\{v_0, \ldots, v_n\}, \rightarrow), \prec$ satisfy Φ, Ψ and Θ,
then $S^-(x) \cap S^+(x) = \emptyset$ and $S(x) \cap T(x) = \emptyset$

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We need to check that

- Φ, Ψ, Θ are satisfied
- $\prec'$ is transitive

If $(\{v_0, \ldots, v_n\}, \rightarrow), \prec$ satisfy Φ, Ψ and Θ, then $S^-(x) \cap S^+(x) = \emptyset$ and $S(x) \cap T(x) = \emptyset$

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We need to check that

- Φ, Ψ, Θ are satisfied
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INPUT $(\{v_n \mid n \in \mathbb{N}\}, \to)$ pseudo-transitive oriented graph

PARTIAL OUTPUT $(\{v_0, \ldots, v_n\}, \prec)$ transitive reorientation

We need to check that

- Φ, Ψ, Θ are satisfied
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INPUT $(\{v_n \mid n \in \mathbb{N}\}, \rightarrow)$ pseudo-transitive oriented graph

PARTIAL OUTPUT $(\{v_0, \ldots, v_n\}, \prec)$ transitive reorientation

$(\{v_0, \ldots, v_n\}, \prec)$ transitive reorientation $\Rightarrow$

$\prec$ satisfy $\Phi, \Psi, \Theta \Rightarrow$

$(\{v_0, \ldots, v_n, x\}, \prec')$ transitive reorientation

We need to check that

- Φ, Ψ, Θ are satisfied
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PARTIAL OUTPUT $(\{v_0, \ldots, v_n\}, \prec)$ transitive reorientation

$(\{v_0, \ldots, v_n\}, \prec)$ transitive reorientation $\Rightarrow$

$\prec$ is **lazy** $\Rightarrow$

$\prec$ satisfy $\Phi, \Psi, \Theta \Rightarrow$

$(\{v_0, \ldots, v_n, x\}, \prec')$ transitive reorientation

If $v_i \rightarrow v_j$ and $v_j \prec v_i$
there exists v_h such that
$v_j \rightarrow v_h \rightarrow v_i$,
$v_j \prec v_h \prec v_i$, and
$h < \min(i, j)$

Complexity

The complexity of the algorithm is $O(|V|^3)$.

The problem of orienting comparability graphs can be solved by an algorithm with complexity $O(\delta \cdot |E|)$, where δ is the maximum degree of a vertex

Complexity

The complexity of a step of the algorithm is $O(|V|^2)$.

- at most $|V^2|$ steps to compute $S_0^+(x)$ and $S_0^-(x)$
- the remaining members of $S^+(x)$ and $S^-(x)$ can be found by a depth-first search algorithm applied to the non-adjacency graph $(V \cup \{x\}, E')$ $(O(|V| + |E|))$
- the rest is linear

The problem of orienting comparability graphs can be solved by an algorithm with complexity $O(\delta \cdot |E|)$, where δ is the maximum degree of a vertex