Coloring Subsets of Euclidean Space

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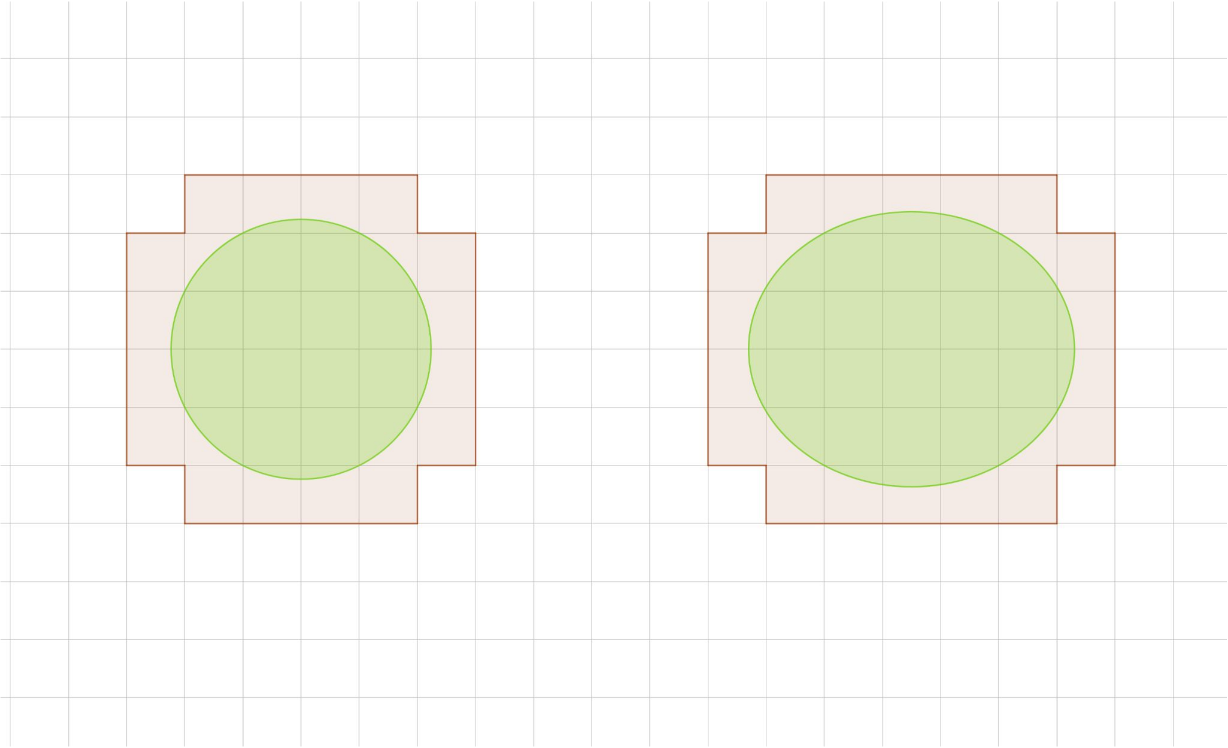
Drawing a set on a screen



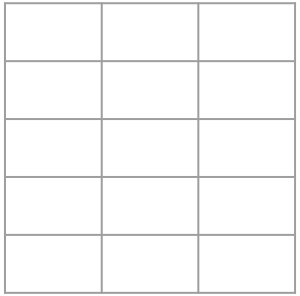
Intuitions

- Two different sets may have a same drawing.
- A set may be drawn on screen in several different ways.
- The drawing doesn’t classify the pixel only as in and out.

Definition: Color set and Resolution

- Color set $\mathcal{C}$ is a subset of integers listing all possible colors.
  - $0 \in \mathcal{C}$
  - Contains at least one positive and one negative integer.
  - Okay to assume $\mathcal{C} = \{-1, 0, 1\}$ for now.

- Resolution $\boldsymbol{r} = (r_1, r_2, \ldots, r_d)$ is a tuple of d positive integers.
  - Grid in $[0, 1]^d$ of equal width $1 / r_i$ for each dimension.
  - *d*-xel: *d*-dimensional version of pixel
  - Volume of *d*-xel is $1 / \prod r_i$.



d = 2, r = (3, 5)

Definition: Color Function

- Color Function
    - $S \subseteq \mathbb{R}^d$, $\boldsymbol{r}$: resolution, P: a d-xel of resolution $\boldsymbol{r}$
    - C_S: $(\boldsymbol{r}, P) \mapsto \mathcal{C}$
    - If $C_S(\boldsymbol{r}, P) > 0$, then $P \cap S \neq \varnothing$
    - If $C_S(\boldsymbol{r}, P) < 0$, then $P \cap S^c \neq \varnothing$
    - If $C_S(\boldsymbol{r}, P) = 0$, then $\exists\, a \in P$, dist$(a, \partial S) < 1 / \min_i r_i$ where dist is the Hausdorff distance

- With this definition, for every set, there always exists a color function which satisfies the condition above.

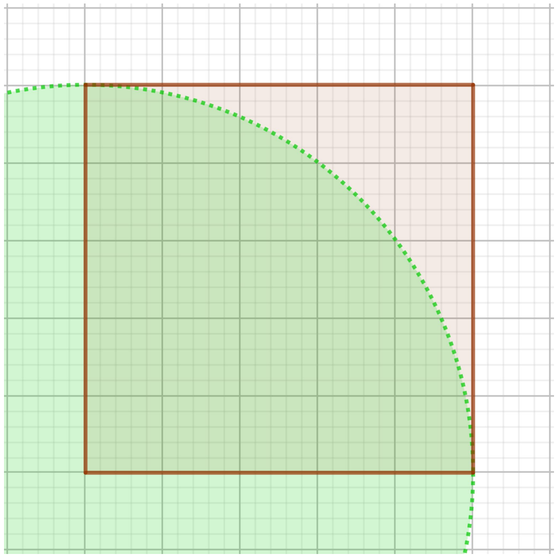
Example: Color Function

$C_S(\boldsymbol{r}, P) > 0 \longrightarrow P \cap S \neq \varnothing$

$C_S(\boldsymbol{r}, P) < 0 \longrightarrow P \cap S^c \neq \varnothing$

$C_S(\boldsymbol{r}, P) = 0 \longrightarrow \exists\, a \in P, \mathrm{dist}(a, \partial S) < 1 / \min_i r_i$

- $x^2 + y^2 < 1$, $\boldsymbol{r} = (5, 5)$



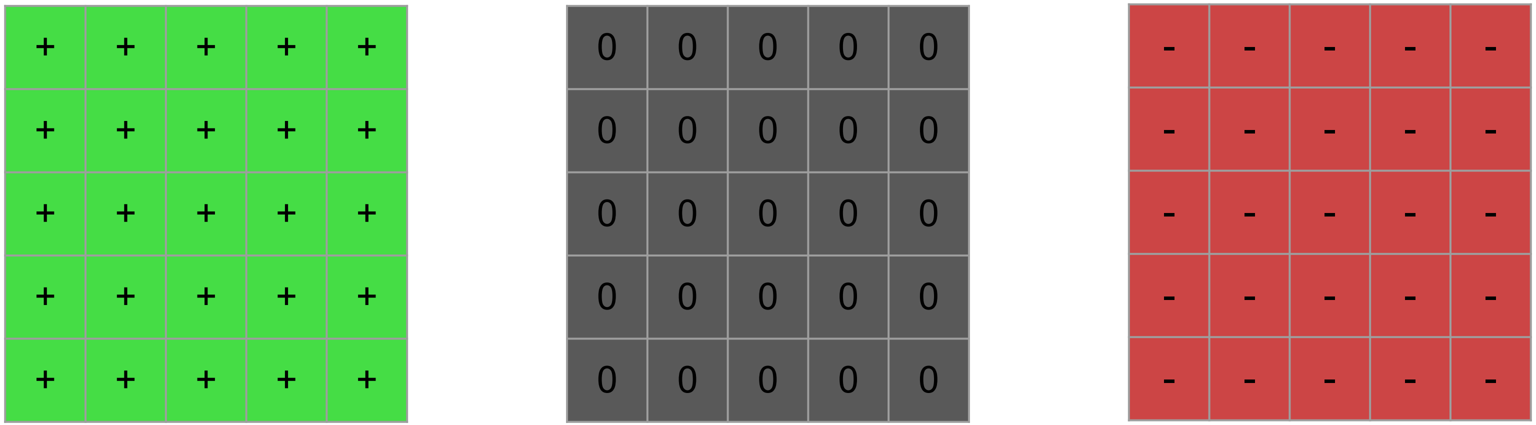
0	0	0	-	-
+	+	+	0	-
+	+	+	+	0
+	+	+	+	0
+	+	+	+	0

0	-	0	-	0
0	0	0	0	-
+	+	+	0	0
+	+	+	0	-
+	+	+	0	0

Colorable Set

- Can we color the set $S = \mathbb{Q}^2 \cap [0, 1]^2$?

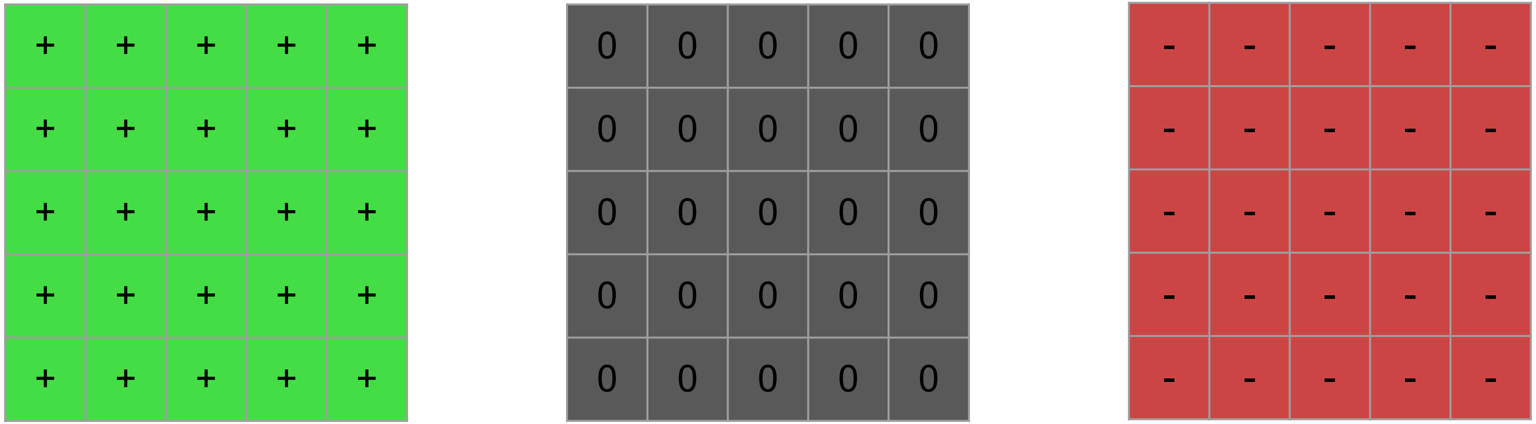
- Since all points of S are boundaries…



Colorable Set

- Can we color the set $S = \mathbb{Q}^2 \cap [0, 1]^2$?

- We don't want to say that S is colorable!
- The result of coloring of a set should 'converge' as $\min_i r_i \to \infty$.



Colorable Set

Theorem 1. Given set $S \subseteq \mathbb{R}^d$, the following are equivalent.

- There is some value v which for every C, $\mathrm{vol}(\{P \mid C_S(\boldsymbol{r}, P) > 0\}) \to v$ as $\min_i r_i \to \infty$.
- There is some value v which for every C, $\mathrm{vol}(\{P \mid C_S(\boldsymbol{r}, P) < 0\}) \to v$ as $\min_i r_i \to \infty$.
- For every C, $\mathrm{vol}(\{P \mid C_S(\boldsymbol{r}, P) = 0\}) \to 0$ as $\min_i r_i \to \infty$.
- $\lambda(\partial S \cap [0, 1]^d) = 0$, where λ is the standard Lebesgue measure of $\mathbb{R}^d$.

Definition: Colorable Set

- A set $S \subseteq \mathbb{R}^d$ is colorable if it satisfies any statement of **Theorem 1**.
    - For every C, $\mathrm{vol}(\{P \mid C_S(\boldsymbol{r}, P) = 0\}) \to 0$ as $\min_i r_i \to \infty$.
    - $\lambda(\partial S) = 0$, where λ is the standard Lebesgue measure of $\mathbb{R}^d$.

- A colorable set S is computably colorable (or drawable) if there is some computable color function C_S.

Properties of Colorable Sets

Let $S \subseteq \mathbb{R}^d$ be a colorable set.

- For any $x \in \mathbb{R}^d$, $\{x\}$ is colorable.
- S^c is colorable.
- ∂S is colorable.
- Finite union of colorable sets is colorable.
  - Countably infinite union of colorable sets may not be colorable. ($\mathbb{Q}$)
- Finite intersection of colorable sets is colorable.
  - Countably infinite intersection of colorable sets may not be colorable. ($\mathbb{R} \setminus \mathbb{Q}$)

Properties of Colorable Sets

A set $S \subseteq \mathbb{R}^d$ is colorable if for every C, $\text{vol}(\{P \mid C_S(\boldsymbol{r}, P) = 0\}) \to 0$ as $\min_i r_i \to \infty$.

- For any $x \in \mathbb{R}^d$, $\{x\}$ is colorable.
  - Color all d-xels negative.

Properties of Colorable Sets

A set $S \subseteq \mathbb{R}^d$ is colorable if for every C, $\text{vol}(\{P \mid C_S(\boldsymbol{r}, P) = 0\}) \to 0$ as $\min_i r_i \to \infty$.

- S^c is colorable.
  - Negate the sign.

0	0	0	-	-
+	+	+	0	-
+	+	+	+	0
+	+	+	+	0
+	+	+	+	0

S

0	0	0	+	+
-	-	-	0	+
-	-	-	-	0
-	-	-	-	0
-	-	-	-	0

S^c

Properties of Colorable Sets

A set $S \subseteq \mathbb{R}^d$ is colorable if for every C, $\mathrm{vol}(\{P \mid C_S(\boldsymbol{r}, P) = 0\}) \to 0$ as $\min_i r_i \to \infty$.

- ∂S is colorable.
  - Replace the color with the minus of the absolute value.

0	0	0	-	-
+	+	+	0	-
+	+	+	+	0
+	+	+	+	0
+	+	+	+	0

S

0	0	0	-	-
-	-	-	0	-
-	-	-	-	0
-	-	-	-	0
-	-	-	-	0

∂S

Properties of Colorable Sets

A set $S \subseteq \mathbb{R}^d$ is colorable if $\lambda(\partial S) = 0$.

- Finite union of colorable sets is colorable.
  - $\partial(A \cup B) \subseteq \partial A \cup \partial B$
  - $\lambda(\partial A) = \lambda(\partial B) = 0 \rightarrow \lambda(\partial(A \cup B)) = 0$
  - Countably infinite union of colorable sets may not be colorable. ($\mathbb{Q}$)

- Finite intersection of colorable sets is colorable.
  - $\partial(A \cap B) \subseteq \partial A \cup \partial B$
  - $\lambda(\partial A) = \lambda(\partial B) = 0 \rightarrow \lambda(\partial(A \cap B)) = 0$
  - Countably infinite intersection of colorable sets may not be colorable. ($\mathbb{R} \setminus \mathbb{Q}$)

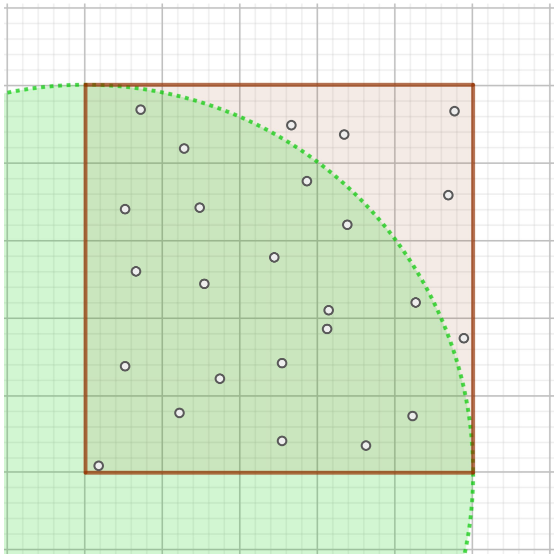
Drawing a Computably Colorable (Drawable) Set

Theorem 2. A colorable set S is computably colorable if both S^0 and $(S^c)^0$ are semidecidable.

- int(A, n, x): Checks if x is in interior of A, up to iteration n.

Drawing a Computably Colorable (Drawable) Set

Theorem 2. A colorable set S is computably colorable if both S^0 and $(S^c)^0$ are semidecidable.



choose any point x in each pixel

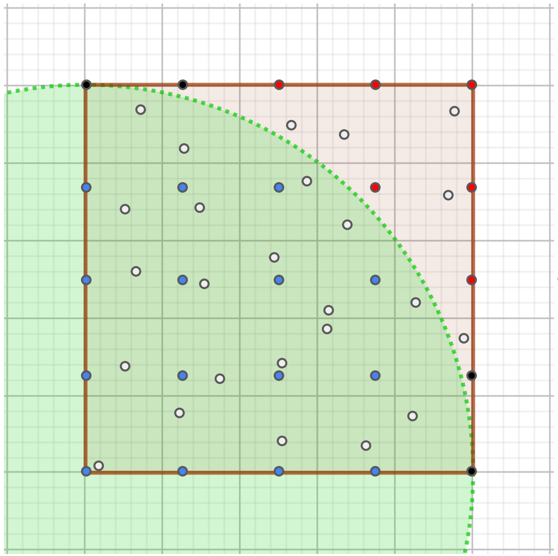
$\text{int}(S, n, x) \rightarrow +$

$\text{int}(S^c, n, x) \rightarrow -$

?	?	?	-	-
+	+	?	?	-
+	+	+	?	?
+	+	+	+	?
+	+	+	+	?

Drawing a Computably Colorable (Drawable) Set

Theorem 2. A colorable set S is computably colorable if both S^0 and $(S^c)^0$ are semidecidable.



point y on a grid
int(S, n, y) → blue
int(S^c, n, y) → red

∃ red point R and blue point B,
dist(R, x) < 1 / r and dist(B, x) < 1 / r
→ 0

?	?	0	-	-
+	+	0	0	-
+	+	+	?	0
+	+	+	+	?
+	+	+	+	?

Drawing a Computably Colorable (Drawable) Set

Limitations of the algorithm:

No pixel is colored positive for a computably colorable set with empty interior.

What can we draw?

- Circle
- Several Simple Geometric Objects
- Cantor Set
- Weierstrass Function
- Koch Snowflake
- Mandelbrot Set, if the hyperbolicity conjecture is true and $\lambda(\partial M) = 0$
- … and much more!

Thank you!