Predicative Bishop-Cheng Measure Theory

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Overview

- The Daniell approach to classical integration and measure theory
- From Daniell spaces to Bishop-Cheng integration spaces
- On the impredicativity of Bishop-Cheng measure theory BCMT
- Predicative Bishop-Cheng measure theory (j.w.w. Max Zeuner)

Other approaches to "computable" measure theory

- Intuitionistic measure theory [11]
- Measure theory within the computability framework of Type-2 Theory of Effectivity [9]
- In Russian constructivism, especially in the work of Šanin [20] and Demuth [5]
- In type theory, where the main interest lies in the creation of probabilistic programming [2]
- In homotopy type theory [10], where HoTT is applied to probabilistic programming

The Daniell approach to classical integration and measure theory

Two approaches to measure and integration theory

- The popular approach: a "from sets to functions"-approach

$$\mu \to \int$$

- The Daniell approach: a "from functions to sets"-approach

$$\int \to \mu$$

Daniell (1918), Weil (1940), Kolmogoroff (1948), Carathéodory (1956), Segal's algebraic integration theory (1954, 1965)

Two approaches to topology

- The popular approach: from open sets to continuous functions

$$(X, \mathcal{T}) \to C(X)$$

- From continuous functions to open sets:

$$C(X) \to (X, \mathcal{T})$$

From open sets to continuous functions in classical and constructive topology

$$\mathcal{T} \to C(X)$$

- topological spaces
- formal spaces
- apartness spaces
- intuitionistic topological spaces
- neighborhood spaces

From continuous functions to open sets in classical and constructive topology

$$C(X) \to \mathcal{T}$$

- limit spaces
- Spanier's quasi-topological spaces
- (Grothendieck) Sites
- Bishop spaces: $F \subseteq \mathbb{F}(X)$ and for every $f \in F$:

$$U(f) := \{x \in X \mid f(x) > 0\}$$

Functions suit better to (classical and) constructive study rather than sets

- The theory of $C(X)$ is used in the study of X.
- To define the characteristic function of a subset we need PEM:

$$\chi_A(x) := \begin{cases} 1 & , x \in A \\ 0 & , x \notin A \end{cases}$$

- In constructive topology the function-theoretic approach of Bishop spaces is shown to be very fruitful.

From measure to integral in classical measure theory

$$\mu \to \int$$

- We start from a measure space $(X, \mathcal{A}, \mu)$
- We define simple functions:

$$\phi = \sum_{i=1}^{n} c_i \chi_{A_i}, \quad \bigcup_{i=1}^{n} A_i = X, \quad A_i \cap A_j = \varnothing$$

- We define measurable functions $f : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ through the Borel sets in $\mathbb{R}$
- A simple function is measurable
- Every positive measurable function is the limit of an increasing sequence $(\phi_n)_{n \in \mathbb{N}}$ of positive, simple functions

From measure to integral in classical measure theory

- The integral of a simple function ϕ is well-defined:

$$\int \phi d\mu := \sum_{i=1}^{n} c_i \mu(A_i)$$

- The integral of a positive measurable function f is well-defined:

$$\int f d\mu := \lim_n \int \phi_n d\mu$$

  f is μ-integrable, if $\int f d\mu \in \mathbb{R}$.

- If f is measurable, then $f_+, f_- \geqslant 0$ are measurable and if f_+, f_- are μ-integrable, let

$$\int f d\mu := \int f_+ d\mu - \int f_- d\mu$$

$$L^1 := \left\{ f : X \to \overline{\mathbb{R}} \mid f \text{ measurable } \& \int |f| d\mu \in \mathbb{R} \right\}$$

The Daniell approach: from integral to measure

$$\int \to \mu$$

- We start from a Daniel space $\left(X, L, \int\right)$, where $L \ni f \mapsto \int f \in \mathbb{R}$
- We extend L and $\int$ using the Bolzano-Weierstrass theorem:

$$L^{+} := \left\{f \in \mathbb{F}(X, \overline{\mathbb{R}}) \mid \exists_{(f_n)_{n=1}^{\infty} \in \mathbb{F}^{+}(\mathbb{N}, L)} \left(f = \lim_n f_n\right)\right\}$$

$$a, b \geqslant 0 \ \& \ f, g \in L^{+} \Rightarrow af + bg \in L^{+}$$

$$\int^{+} : L^{+} \to \overline{\mathbb{R}}$$

$$\int^{+} f = \lim_n \int f_n \in \overline{\mathbb{R}}$$

- If $f: X \to \overline{\mathbb{R}}$, let

$$\overline{\int} f := \inf \left\{ \int^{+} g \in \mathbb{R} \mid g \in L^{+} \ \& \ g \geqslant f \right\}$$

$$\underline{\int} f := \sup \left\{ \int^{+} h \in \mathbb{R} \mid h \in L^{+} \ \& \ h \leqslant f \right\}$$

If $f \in L^{+}$, then clearly

$$\overline{\int} f = \underline{\int} f = \int^{+} f.$$

- A function $f: X \to \overline{\mathbb{R}}$ is called integrable, if

$$\overline{\int} f = \underline{\int} f \quad \& \quad \overline{\int} f \in \mathbb{R}.$$

- L^1 is the set of integrable functions and if $f \in L^1$,

$$\int^{1} f := \overline{\int} f$$

The proof of the following theorem is classical, as it uses PEM:

$$\lim_n \int f_n = +\infty \ \vee \ \lim_n \int f_n \in \mathbb{R}.$$

Theorem (CLASS)

If $\mathcal{D} := \big(X, L, \int\big)$ *is a Daniell space, then* $\mathcal{D}^1 := \big(X, L^1, \int^1\big)$ *is a Daniell space that extends* $\mathcal{D}$ *i.e.,* $L \subseteq L_1$ *and*

$$\int^1 f = \int f,$$

for every $f \in L$*. Moreover, if* $(f_n)_{n=1}^{\infty}$ *is an increasing sequence in* L^1*, and* $f : X \to \overline{\mathbb{R}}$ *such that* $f = \lim_n f_n$*, then*

$$f \in L^1 \Leftrightarrow \lim_n \int^1 f_n \in \mathbb{R}.$$

$$\int^1 f = \lim_n \int^1 f_n.$$

Definition

Let $\mathcal{D} := \left(X, L, \int\right)$ be a Daniell space. A function $f : X \to [0, +\infty]$ is called measurable,

$$\forall_{g \in L^1} \left(f \wedge g \in L^1\right).$$

$A \subseteq X$ is measurable, if χ_A is measurable.

A is integrable, if $\chi_A \in L^1$.

Let $\mathcal{A}$ be the set of all measurable sets.

The formulation and the proof of the next theorem, which connects Daniell integration to standard measure integration, relies on PEM.

Theorem (Stone, CLASS)

Let a Daniel space $\mathcal{D}^1 := \left(X, L^1, \int^1\right)$*. If* $X \in \mathcal{A}$*, then* $(X, \mathcal{A}, \mu)$ *is a measure space, where for every* $A \in \mathcal{A}$

$$\mu(A) := \begin{cases} \int^1 \chi_A & , A \text{ is integrable} \\ +\infty & , \text{otherwise.} \end{cases}$$

Moreover, $f \in L^1$ *if and only if* f *is* μ*-integrable, and then*

$$\int^1 f = \int f d\mu.$$

Integral space

$X \neq \emptyset$, L a Riesz space in $\mathbb{F}(X)$, and

$$\int : L \to \mathbb{R}$$

ia a positive, linear functional, continuous under monotone limits:

$(\mathtt{IntLin})$ $\int(af + bg) = a\int f + b\int g$, for every $a, b \in \mathbb{R}$ and $f, g \in L$.

$(\mathtt{IntPos})$ $f \geqslant 0 \Rightarrow \int f \geqslant 0$, for every $f \in F$.

$(\mathtt{IntCont})$ For every sequence $(f_n)_{n=1}^{\infty}$ in L with $f_n \geqslant f_{n+1}$,

$$\lim_n f_n = 0 \Rightarrow \lim_n \int f_n = 0.$$

$\int$ is called an integral on L and $\mathcal{I} := \left(X, L, \int\right)$ an integral space.

Daniell space

- A positive, linear functional $\int$ on L is a Daniell integral, if (IntDan) for every $(f_n)_{n=1}^{\infty}$ in L, $f_n \leqslant f_{n+1}$, and $f \in L$,

$$f \leqslant \lim_n f_n \Rightarrow \int f \leqslant \lim_n \int f_n.$$

  In this case $\mathcal{D} := \left(X, L, \int\right)$ is a Daniell space.
- An integral $\int$ on L is called complete, if (IntCompl) for every $(f_n)_{n=1}^{\infty}$ in L, $f_n \leqslant f_{n+1}$, and $f \in \mathbb{F}(X)$,

$$\lim_n f_n = f \Rightarrow f \in L \; \& \; \lim_n \int f_n = \int f.$$

  In this case $\mathcal{I} := \left(X, L, \int\right)$ is complete.
- If $\mathcal{I}$ is complete, and $1 \in L$ such that $\int 1 = 1$, then $\mathcal{I}$ is a probability Daniell space.

If $\mathcal{R}[0,1]$ is the set of Riemann-integrable functions on $[0,1]$ and $\int_{\mathcal{R}}$ is the Riemann integral on $\mathcal{R}$, then $\left([0,1], \mathcal{R}[0,1], \int_{\mathcal{R}}\right)$ is an integral space, which is not complete;

If $(q_n)_{n=1}^{\infty}$ is a fixed enumeration of $\mathbb{Q}$, and

$$f_n(x) := \begin{cases} 1 & , x \in \{q_1, \ldots, q_n\} \\ 0 & , \text{otherwise,} \end{cases}$$

then $\lim f_n$ is the Dirichlet function, which is not in $\mathcal{R}[0,1]$.

To get the completeness property for $\int_{\mathcal{R}}$ on $[0,1]$ it suffices that $(f_n)_{n=1}^{\infty}$ converges uniformly to f

- If $C(M)$ is the set of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$ which are 0 outside $[-M, M]$, for some $M > 0$, and

$$\int_M f := \int_{\mathcal{R}} f := \int_{-\infty}^{+\infty} f(x)dx,$$

then $\left(\mathbb{R}, C(M), \int_{\mathcal{R}}\right)$ is a Daniell space.

- If $C^{\mathrm{supp}}(\mathbb{R}^n)$ is the set of continuous real-valued functions with compact support i.e., the closure of

$$[f \neq 0] := \{x \in \mathbb{R}^n \mid f(x) \neq 0\}$$

is compact, and if K is compact with $[f \neq 0] \subseteq K$, let

$$\int f := \int_K f(x)dx := \int_{\mathcal{R}} f\chi_K.$$

Then $\mathcal{D} := \left(\mathbb{R}^n, C^{\mathrm{supp}}(\mathbb{R}^n), \int\right)$ is a Daniell space and its completion $\mathcal{D}^1$ is the Lebesgue (Daniell) space.

- If X is a locally compact Hausdorff space and $L = C^{\mathrm{supp}}(X)$, then every positive, linear functional on L is an integral.

- If $(X, \mathcal{A}, \mu)$ is a σ-finite measure space, $L(\mu)$ is the set of μ-integrable functions from X to $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ and

$$\int_\mu f := \int f d\mu,$$

for every $f \in L(\mu)$, then $\left(X, L(\mu), \int_\mu\right)$ is a Daniell space.

From Daniell spaces to Bishop-Cheng integration spaces

BISH: $(\texttt{IntCont})$ is equivalent to the following condition:

$(\texttt{IntCont}')$ For every monotone sequence $(f_n)_{n=1}^{\infty}$ in L such that $f(x) = \lim_n f_n(x) \in \mathbb{R}$, for every $x \in X$, and $f \in L$,

$$\int f = \int \lim_n f_n = \lim_n \int f_n.$$

CLASS: $(\texttt{IntDan}) \Rightarrow (\texttt{IntCont}')$

BISH: A positive, linear functional $\int$ on L is a Daniell integral, if and only if

$(\texttt{IntDan}')$ for every sequence $(f_n)_{n=1}^{\infty}$ in L with $f_n \geqslant 0$, for every $n \in \mathbb{N}^+$, and for every $f \in L$,

$$f \leqslant \sum_{n=1}^{\infty} f_n \Rightarrow \int f \leqslant \sum_{n=1}^{\infty} \int f_n.$$

Classically this implication is equivalent to

$$\neg\left(\int f \leqslant \sum_{n=1}^{\infty} \int f_n\right) \Rightarrow \neg\left(f \leqslant \sum_{n=1}^{\infty} f_n\right),$$

$(\texttt{IntBishCheng})$

$$\left[\sum_{n=1}^{\infty} \int f_n \in \mathbb{R} \ \& \ \int f > \sum_{n=1}^{\infty} \int f_n\right] \Rightarrow$$

$$\exists_{x \in X}\left(\sum_{n=1}^{\infty} f_n(x) \in \mathbb{R} \ \& \ f(x) > \sum_{n=1}^{\infty} f_n(x)\right)$$

BISH: $(\texttt{IntBishCheng}) \Rightarrow (\texttt{IntCont}')$,
provided that $\lim_n \int f_n$ exists in the hypothesis of $(\texttt{IntCont}')$

A Riesz space L in $\mathbb{F}(X)$ satisfies the Stone condition, $\texttt{Stone}(L)$, if

$$(\texttt{Stone}) \qquad \forall_{f \in L}\big(f \wedge 1 \in L\big).$$

Used by Stone [25] to prove the integrability of $[f > a]$, $f \in L$

If $1 \in L$, then $\texttt{Stone}(L)$

$\texttt{Stone}\big(C^{\mathrm{supp}}(\mathbb{R}^n)\big)$, as $[f \neq 0] = [(f \wedge 1) \neq 0]$, and $1 \notin C^{\mathrm{supp}}(\mathbb{R}^n)$.

$$f \vee (-1) = -\big[(-f) \wedge 1\big] \in L$$

If $a \neq 0$, then

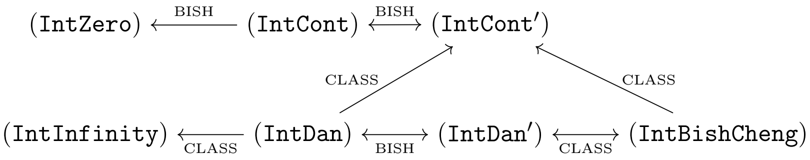
$$f \wedge a = a\left[\left(\frac{1}{a}f\right) \wedge 1\right] \in L,$$

$$(\texttt{IntInfinity}) \qquad \lim_n \int (f \wedge n) = \int f,$$

$$(\texttt{IntZero}) \qquad \lim_n \int \left(|f| \wedge \frac{1}{n} \right) = 0.$$

(CLASS): $(\texttt{IntDan}) \Rightarrow (\texttt{IntInfinity})$

(BISH): $(\texttt{IntCont}) \Rightarrow (\texttt{IntZero})$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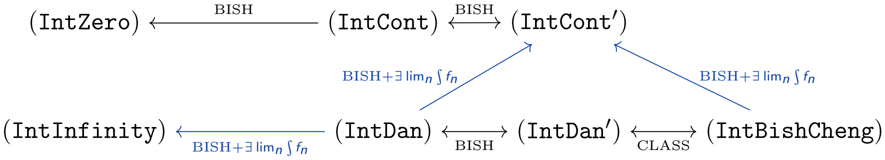
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CLASS
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The integral $\int$ is non-trivial, if there is $f \in L$ s.t. $\int f \neq 0$.

BISH: The following are equivalent:

(i) *There is* $f \in L$ *such that* $\int f \neq_{\mathbb{R}} 0$.

(ii) *For every* $a \in \mathbb{R}$ *there is* $f_a \in L$ *such that* $\int f_a = a$.

(iii) *There is* $u \in L$ *such that* $\int u = 1$.

(iv) *There is* $u^* \in L$ *such that* $u^* \geqslant 0$ *and* $\int u^* = 1$.

(v) *For every* $a \in \mathbb{R}$ *there is* $f_a^* \in L$ *such that* $f_a^* \geqslant 0$ *and* $\int f_a^* = a$.

A Bishop-Cheng integration space:

$(X, L, \int)$, where $(X, =_X, \neq_X)$ is inhabited,

L is a subset of the set $\mathfrak{F}^{se}(X)$ of strongly extensional, real-valued partial functions

$\int : L \to \mathbb{R}$, s.t.

- If $f, g \in L$ and $\alpha, \beta \in \mathbb{R}$, then $\alpha f + \beta g$, $|f|$, and $f \wedge 1$ are in L
- $\int$ is linear
- There is $u \in L$ s.t. $\int u = 1$
- $(\texttt{IntBishCheng})$ and $(\texttt{IntInfinity})$ and $(\texttt{IntZero})$

Why partiality?

A key feature of the Daniell approach is that the transition form functions to sets requires the use of characteristic functions:

$A \subseteq X$ is measurable, if χ_A is measurable.

A is integrable, if $\chi_A \in L^1$

To curry out this constructively one has to use partial functions

$$X \rightharpoonup 2$$

If partial functions $X \rightharpoonup 2$ are in $L(L^1)$, then L has to be a set of real-valued, partial functions on X, and

A is not a subset of X, but a complemented subset of X w.r.t. a given inequality $\neq$ on X.

Let X be a set. An inequality on X, or an apartness relation on X, is a relation $x \neq_X y$ s.t. the following conditions are satisfied:

(Ap_1) $\forall_{x,y \in X}\big(x =_X y \ \& \ x \neq_X y \Rightarrow \bot\big)$.
(Ap_2) $\forall_{x,y \in X}\big(x \neq_X y \Rightarrow y \neq_X x\big)$.
(Ap_3) $\forall_{x,y \in X}\big(x \neq_X y \Rightarrow \forall_{z \in X}(z \neq_X x \ \vee \ z \neq_X y)\big)$.

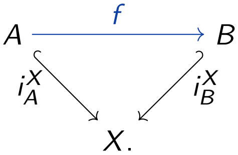
If $(X, =_x, \neq_X)$ and $(Y, =_Y, \neq_Y)$ a function $f \colon X \to Y$ is strongly extensional, if for every $x, x' \in X$

$$f(x) \neq_Y f(x') \Rightarrow x \neq_X x'.$$

Subsets in Bishop set theory

Let X be a set. A subset of X is a pair (A, i_A^X), where A is a set and $\imath_A^X : A \hookrightarrow X$ is an embedding of A into X.

If (A, i_A^X) and (B, i_B^X) are subsets of X, then A is a subset of B, if there is $f : A \to B$ s.t. tfdc



The totality of the subsets of X is the powerset $\mathcal{P}(X)$ of X, and it is equipped with the equality

$$(A, i_A^X) =_{\mathcal{P}(X)} (B, i_B^X) :\Leftrightarrow A \subseteq B \ \& \ B \subseteq A.$$

$\mathcal{P}(X)$ is a class (not a set).

If $(A, i_A^X), (B, i_B^X) \subseteq X$, their union $A \cup B$ is the totality defined by

$$z \in A \cup B :\Leftrightarrow z \in A \ \vee \ z \in B,$$

equipped with the non-dependent assignment routine $i_{A\cup B}^X : A \cup B \rightsquigarrow X$, defined by

$$i_{A\cup B}^X(z) := \begin{cases} i_A^X(z) & , z \in A \\ i_B^X(z) & , z \in B. \end{cases}$$

If $z, w \in A \cup B$, we define

$$z =_{A\cup B} w :\Leftrightarrow i_{A\cup B}^X(z) =_X i_{A\cup B}^X(w)$$

$$z \neq_{A\cup B} w :\Leftrightarrow i_{A\cup B}^X(z) \neq_X i_{A\cup B}^X(w)$$

If $(A, i_A^X), (B, i_B^X) \subseteq X$, their intersection $A \cap B$ is the totality defined by separation on $A \times B$ as follows:

$$A \cap B := \{(a, b) \in A \times B \mid i_A^X(a) =_X i_B^X(b)\}.$$

Let the non-dependent assignment routine $i_{A \cap B}^X : A \cap B \rightsquigarrow X$,

$$i_{A \cap B}^X(a, b) := i_A^X(a),$$

for every $(a, b) \in A \cap B$. If (a, b) and (a', b') are in $A \cap B$, let

$$(a, b) =_{A \cap B} (a', b') :\Leftrightarrow i_{A \cap B}^X(a, b) =_X i_{A \cap B}^X(a', b') :\Leftrightarrow i_A^X(a) =_X i_A^X(a')$$

$$(a, b) \neq_{A \cap B} (a', b') :\Leftrightarrow i_A^X(a) \neq_X i_A^X(a')$$

A complemented subset of a set $(X, =_X, \neq_X)$ is a quadruple

$$\boldsymbol{A} := (A^1, i^X_{A^1}, A^0, i^X_{A^0}),$$

or simply $\boldsymbol{A} := (A^1, A^0)$, where $(A^1, i^X_{A^1})$ and $(A^0, i^X_{A^0}) \subseteq X$ s.t.

$$A^1]\![A^0 :\Leftrightarrow \forall_{a^1 \in A^1} \forall_{a^0 \in A^0} \big(i^X_{A^1}(a^1) \neq_X i^X_{A^0}(a^0) \big).$$

$$\boldsymbol{A} \subseteq \boldsymbol{B} :\Leftrightarrow A^1 \subseteq B^1 \ \& \ B^0 \subseteq A^0,$$

Let $\mathcal{P}^{]\![}(X)$ be their totality, equipped with the equality

$$\boldsymbol{A} =_{\mathcal{P}^{]\![}(X)} \boldsymbol{B} :\Leftrightarrow \boldsymbol{A} \subseteq \boldsymbol{B} \ \& \ \boldsymbol{B} \subseteq \boldsymbol{A}.$$

$\mathcal{P}^{]\![}(X)$ is a proper class.

$$x \in \boldsymbol{A} :\Leftrightarrow x \in A^1 \quad \& \quad x \notin \boldsymbol{A} :\Leftrightarrow x \in A^0$$

If $\mathrm{dom}(\boldsymbol{A}) := A^1 \cup A^0$ is the domain of $\boldsymbol{A}$, the indicator function of a $\boldsymbol{A}$, or its characteristic function, is the assignment routine

$$\chi_{\boldsymbol{A}} : \mathrm{dom}(\boldsymbol{A}) \rightsquigarrow 2$$

$$\chi_{\boldsymbol{A}}(x) := \begin{cases} 1 & , x \in A^1 \\ 0 & , x \in A^0 \end{cases}$$

$\chi_{\boldsymbol{A}}$ is a strongly extensional function

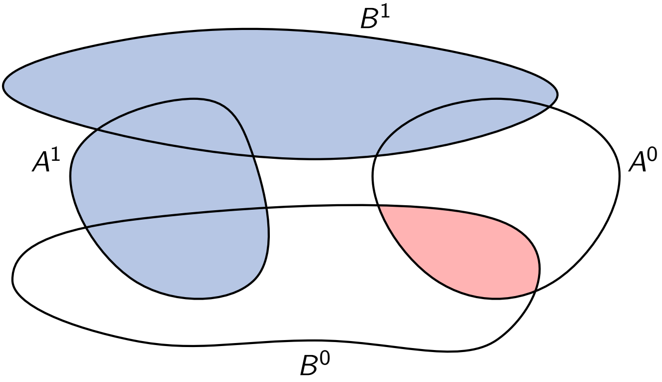
First kind of operations on complemented subsets

$$-\boldsymbol{A} := (A^0, A^1),$$

$$\boldsymbol{A} \cup \boldsymbol{B} := (A^1 \cup B^1, A^0 \cap B^0),$$

$$x \in \boldsymbol{A} \cup \boldsymbol{B} \Leftrightarrow x \in \boldsymbol{A} \ \vee \ x \in \boldsymbol{B} :\Leftrightarrow x \in A^1 \ \vee \ x \in B^1 :\Leftrightarrow x \in A^1 \cup B^1$$

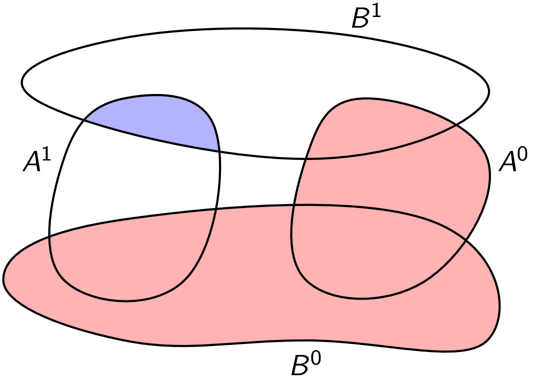
$$x \notin \boldsymbol{A} \cup \boldsymbol{B} \Leftrightarrow x \notin \boldsymbol{A} \ \& \ x \notin \boldsymbol{B} :\Leftrightarrow x \in A^0 \ \& \ x \in B^0 :\Leftrightarrow x \in A^0 \cap B^0$$



$$\boldsymbol{A} \cap \boldsymbol{B} := (A^1 \cap B^1, A^0 \cup B^0),$$

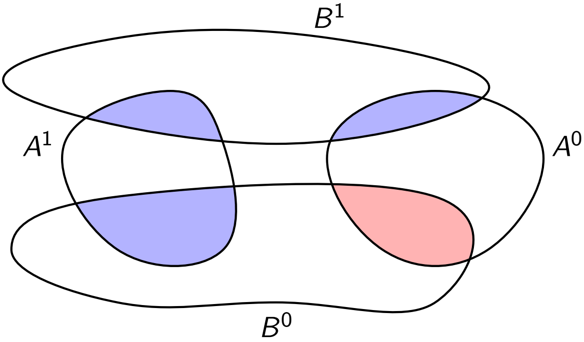
$$x \in \boldsymbol{A} \cap \boldsymbol{B} \Leftrightarrow x \in \boldsymbol{A} \ \& \ x \in \boldsymbol{B} :\Leftrightarrow x \in A^1 \ \& \ x \in B^1 :\Leftrightarrow x \in A^1 \cap B^1$$

$$x \notin \boldsymbol{A} \cap \boldsymbol{B} \Leftrightarrow x \notin \boldsymbol{A} \ \vee \ x \notin \boldsymbol{B} :\Leftrightarrow x \in A^0 \ \vee \ x \in B^0 :\Leftrightarrow x \in A^0 \cup B^0$$



Second kind of operations on complemented subsets

$$\boldsymbol{A} \vee \boldsymbol{B} := \big([A^1 \cap B^1] \cup [A^1 \cap B^0] \cup [A^0 \cap B^1],\ A^0 \cap B^0\big)$$



The multiplicative version P `par` Q of $P \vee Q$ in linear logic:

"if not P, then Q; and if not Q, then P".

$$x \in \boldsymbol{A} \vee \boldsymbol{B} :\Leftrightarrow [x \notin \boldsymbol{A} \Rightarrow x \in \boldsymbol{B}] \ \& \ [x \notin \boldsymbol{B} \Rightarrow x \in \boldsymbol{A}].$$

By Ex falsum quodlibet the implication $x \notin \boldsymbol{A} \Rightarrow x \in \boldsymbol{B}$ holds if $x \in \boldsymbol{A} :\Leftrightarrow x \in A^1$, or if $x \notin \boldsymbol{A} :\Leftrightarrow x \in A^0$ and $x \in \boldsymbol{B} :\Leftrightarrow x \in B^1$ i.e., if $x \in A^0 \cap B^1$. Hence, the first implication holds if $x \in A^1 \cup (A^0 \cap B^1)$. Similarly, the second holds if $x \in B^1 \cup (B^0 \cap A^1)$. Thus

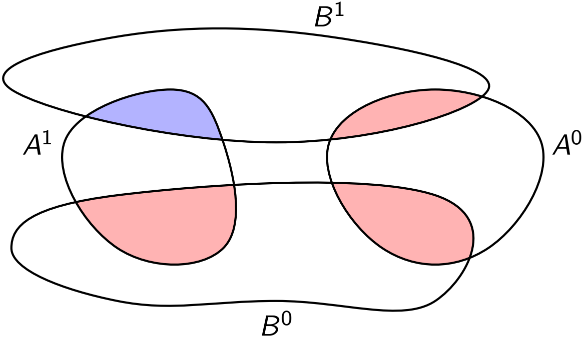
$$x \in \boldsymbol{A} \vee \boldsymbol{B} \Leftrightarrow x \in [A^1 \cup (A^0 \cap B^1)] \cap [B^1 \cup (B^0 \cap A^1)],$$

and the last intersection is equal to $(\boldsymbol{A} \vee \boldsymbol{B})^1$.
One then defines $x \notin \boldsymbol{A} \vee \boldsymbol{B} :\Leftrightarrow x \notin \boldsymbol{A} \ \& \ x \notin \boldsymbol{B}$

$$\boldsymbol{A} \wedge \boldsymbol{B} := \left(A^1 \cap B^1,\ [A^1 \cap B^0] \cup [A^0 \cap B^1] \cup [A^0 \cap B^0]\right)$$

$$x \in \boldsymbol{A} \wedge \boldsymbol{B} :\Leftrightarrow x \in \boldsymbol{A} \ \& \ x \in \boldsymbol{B}$$

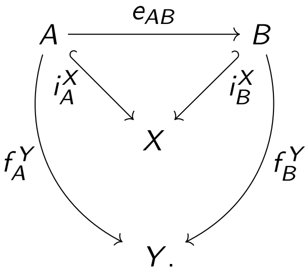
$$x \notin \boldsymbol{A} \wedge \boldsymbol{B} :\Leftrightarrow x \in (-\boldsymbol{A}) \vee (-\boldsymbol{B})$$



- Shulman [23]: Bishop's complemented subsets correspond roughly to the Chu construction
- P. [16]: there is a full embedding of $\mathcal{P}^{\mathbb{II}}(X)$ into $\mathbf{Chu}(\mathbf{Set}, X \times X)$
- The Chu construction is a method of generating a $*$-autonomous category from a closed symmetric monoidal category $\mathcal{C}$ and some $\gamma \in C_0$
- $*$-autonomous categories provide models for classical (multiplicative) linear logic
- The abstract lattice version of $\mathcal{P}^{\mathbb{II}}(X)$ is not a Heyting algebra, it is what can be called a Bishop algebra.

A partial function $f : X \rightharpoonup Y$ is

a triplet (A, i_A^X, f_A^Y), where $(A, i_A^X) \subseteq X$, and $f_A^Y \in \mathbb{F}(A, Y)$. $(A, i_A^X, f_A^Y) \leqslant (B, i_B^X, f_B^Y)$, or simpler $f_A^Y \leqslant f_B^Y$, if there is $e_{AB} : A \to B$ such that the following inner diagrams commute



The totality of partial functions from X to Y is the partial function space $\mathfrak{F}(X, Y)$, and it is equipped with the equality

$$(A, i_A^X, f_A^Y) =_{\mathfrak{F}(X,Y)} (B, i_B^X, f_B^Y) :\Leftrightarrow f_A^Y \leqslant f_B^Y \ \& \ f_B^Y \leqslant f_A^Y.$$

The advantage of the second kind of operations

$$\chi_{\boldsymbol{A} \vee \boldsymbol{B}} =_{\mathfrak{F}(X,2)} \chi_{\boldsymbol{A}} \vee \chi_{\boldsymbol{B}}$$

$$\begin{aligned} \mathrm{dom}(\boldsymbol{A}) \cap \mathrm{dom}(\boldsymbol{B}) &= \left[A^1 \cup A^0\right] \cap \left[B^1 \cup B^0\right] \\ &= \left[A^1 \cap B^1\right] \cup \left[A^1 \cap B^0\right] \cap \left[A^0 \cup B^1\right] \cup \left[A^0 \cap B^0\right] \end{aligned}$$

$$\chi_{\boldsymbol{A} \wedge \boldsymbol{B}} =_{\mathfrak{F}(X,2)} \chi_{\boldsymbol{A}} \cdot \chi_{\boldsymbol{B}}$$

Why strong extensionality for the elements of L?

If $(X, =_X, \neq_X)$, let the proper class-assignment routines

$$\chi^X : \mathcal{P}^{\rrbracket\llbracket}(X) \rightsquigarrow \mathfrak{F}^{\mathrm{se}}(X, 2), \qquad \boldsymbol{A} \mapsto \chi^X(\boldsymbol{A}) =: \chi_{\boldsymbol{A}}$$

$$\chi_{\boldsymbol{A}} := \big(A^1 \cup A^0, i^X_{A^1 \cup A^0}, \chi^2_{A^1 \cup A^0}\big),$$

$$\delta^X : \mathfrak{F}^{\mathrm{se}}(X, 2) \rightsquigarrow \mathcal{P}^{\rrbracket\llbracket}(X), \qquad f_A := \big(A, i^X_A, f^2_A\big) \mapsto \boldsymbol{\delta}^X(f_A)$$

$$\boldsymbol{\delta}^X(f_A) := \Bigg(\delta^1_0(f^2_A),\ (i^X_A)_{|\delta^1_0(f^2_A)},\ \delta^0_0(f^2_A),\ (i^X_A)_{|\delta^0_0(f^2_A)}\Bigg),$$

where

$$\delta^1_0(f^2_A) := \big\{a \in A \mid f^2_A(a) =_2 1\big\} =: [f^2_A =_2 1],$$

$$\delta^0_0(f^2_A) := \big\{a \in A \mid f^2_A(a) =_2 0\big\} =: [f^2_A =_2 0],$$

(i) χ^X *is a well-defined, proper class-function.*

(ii) δ^X *is a well-defined, proper class-function.*

(iii) χ^X *and* δ^X *are inverse to each other.*

On the impredicativity of Bishop-Cheng measure theory

$\mathfrak{F}(X, \mathbb{R})$ is a proper class

$\mathfrak{F}^{\mathrm{se}}(X, \mathbb{R})$ is defined by class-separation, and it is also a proper class

L is considered as a subset of (the set) $\mathfrak{F}^{\mathrm{se}}(X, \mathbb{R})$

As Spitters remarks: the original Bishop-Cheng measure theory is impredicative, therefore hard to implement to some functional programming language.

This is why later attempts to measure theory within BISH, like these of Coquand and Palmgren [7], Spitters [24] and Ishihara [12], are developed in an abstract setting that involves certain boolean rings or vector lattices.

Impredicative definition of L^1 in BCMT

$f \in \mathfrak{F}^{se}(X)$ is integrable, if there is $(f_n)_{n=1}^{\infty} \subseteq L$ s.t.

$$\sum_{n=1}^{\infty} \int |f_n| \in \mathbb{R}$$

and for every $x \in X$

$$\sum_{n=1}^{\infty} |f_n(x)| \in \mathbb{R} \Rightarrow f(x) = \sum_{n=1}^{\infty} f_n(x).$$

$$L^1 := \{f \in \mathfrak{F}^{se}(X) \mid f \text{ integrable}\}$$

The separation scheme on a proper class does not define, in general, a set. L^1 is a set, only if $\mathfrak{F}^{se}(X)$ is considered to be a set. This is not predicatively correct, as the membership condition of $\mathfrak{F}^{se}(X)$, or of $\mathfrak{F}(X)$, requires quantification over the universe of sets $\mathbb{V}_0$.

Many repercussions of the impredicativity of L^1

$F \subseteq X$ is full, if $\exists_{f \in L^1} \big(\mathrm{dom}(f) \subseteq F\big)$

$f =_{\texttt{ae}} g$ iff there is $F \subseteq X$ full such that $F \subseteq \mathrm{dom}(f) \cap \mathrm{dom}(g)$ and $f_{|F} = g_{|F}$

Quantification over the proper class $\mathcal{P}^{\mathbb{II}}(X)$ in the definition of measure space within BCMT.

Predicative Bishop-Cheng measure theory

A family of sets indexed by a set I

$\Lambda := (\lambda_0, \lambda_1)$, where $\lambda_0 : I \rightsquigarrow \mathbb{V}_0$, and

$$\lambda_1 : \bigcurlywedge_{(i,j) \in D(I)} \mathbb{F}\big(\lambda_0(i), \lambda_0(j)\big), \quad \lambda_1(i,j) := \lambda_{ij}, \quad (i,j) \in D(I),$$

(a) For every $i \in I$, we have that $\lambda_{ii} := \mathrm{id}_{\lambda_0(i)}$.

(b) If $i =_I j$ and $j =_I k$, the following diagram commutes

$$\begin{array}{ccc} \lambda_0(i) & & \\ {\scriptstyle \lambda_{ij}}\downarrow & \searrow^{\lambda_{ik}} & \\ \lambda_0(j) & \xrightarrow[\lambda_{jk}]{} & \lambda_0(k). \end{array}$$

If $i =_I j$, we call the function λ_{ij} the transport map from $\lambda_0(i)$ to $\lambda_0(j)$. We call the assignment routine λ_1 the modulus of function-likeness of λ_0. An I-family of sets is an an I-*set* of sets, if

$$\forall_{i,j \in I}\big(\lambda_0(i) =_{\mathbb{V}_0} \lambda_0(j) \Rightarrow i =_I j\big).$$

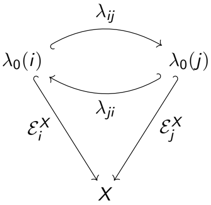
A family of subset of X indexed by a set I

$\Lambda(X) := (\lambda_0, \mathcal{E}^X, \lambda_1)$, where $\lambda_0 : I \rightsquigarrow \mathbb{V}_0$,

$$\mathcal{E}^X : \bigcurlywedge_{i \in I} \mathbb{F}\big(\lambda_0(i), X\big), \quad \mathcal{E}^X(i) := \mathcal{E}_i^X; \quad i \in I,$$

$$\lambda_1 : \bigcurlywedge_{(i,j) \in D(I)} \mathbb{F}\big(\lambda_0(i), \lambda_0(j)\big), \quad \lambda_1(i,j) := \lambda_{ij}; \quad (i,j) \in D(I),$$

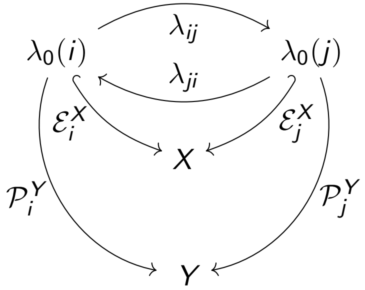
(a) *For every $i \in I$, the function $\mathcal{E}_i^X : \lambda_0(i) \to X$ is an embedding.*
(b) *For every $i \in I$, we have that $\lambda_{ii} := \mathrm{id}_{\lambda_0(i)}$.* (c) *For every $(i,j) \in D(I)$: $\mathcal{E}_i^X = \mathcal{E}_j^X \circ \lambda_{ij}$ and $\mathcal{E}_j^X = \mathcal{E}_i^X \circ \lambda_{ji}$*



$\mathcal{E}^X$ is a modulus of embeddings for λ_0, and λ_1 a modulus of transport maps for λ_0.

Family of partial functions $X \rightharpoonup Y$ over I

$\Lambda(X, Y) := (\lambda_0, \mathcal{E}^X, \lambda_1, \mathcal{P}^Y)$, where
$\Lambda(X) := (\lambda_0, \mathcal{E}^X, \lambda_1) \in \mathtt{Fam}(I, X)$ and $\mathcal{P}^Y : \bigcurlywedge_{i \in I} \mathbb{F}\big(\lambda_0(i), Y\big)$ with
$\mathcal{P}^Y(i) := \mathcal{P}_i^Y$, for every $i \in I$,



We call $\mathcal{P}^Y$ a modulus of partial functions for λ_0, and $\Lambda(X)$ the I-family of domains of $\Lambda(X, Y)$

Integration space within BST

Let $(X, =_X, \neq_X)$ be an inhabited set

$$\Lambda(X, \mathbb{R}) := (\lambda_0, \mathcal{E}^X, \lambda_1, \mathcal{P}^{\mathbb{R}}) \in \mathtt{Fam}(I, X, \mathbb{R})$$

$$f_i := \big(\lambda_0(i), \mathcal{E}_i^X, f_i^{\mathbb{R}}\big)$$

is strongly extensional, for every $i \in I$

$\lambda_0 I(X, \mathbb{R})$ be the totality I, equipped with the equality

$$i =_{\lambda_0 I(X,\mathbb{R})} j :\Leftrightarrow f_i =_{\mathcal{F}(X)} f_j$$

$$\int : \lambda_0 I(X, \mathbb{R}) \to \mathbb{R},$$

$$f_i \mapsto \int f_i; \qquad i \in I,$$

such that the following conditions hold:

Integration space

$$\forall_{i\in I}\forall_{a\in\mathbb{R}}\exists_{j\in I}\left(af_i =_{\mathfrak{F}(X)} f_j \ \ \& \ \ \int f_j =_{\mathbb{R}} a\int f_i\right)$$

$$\forall_{i,j\in I}\exists_{k\in I}\left(f_i + f_j =_{\mathfrak{F}(X)} f_k \ \ \& \ \ \int f_k =_{\mathbb{R}} \int f_i + \int f_j\right)$$

$$\forall_{i\in I}\exists_{j\in I}\left(|f_i| =_{\mathfrak{F}(X)} f_j\right)$$

$$\forall_{i\in I}\exists_{j\in I}\left(f_i \wedge 1 =_{\mathfrak{F}(X)} f_j\right)$$

$$\exists_{i\in I}\left(\int f_i =_{\mathbb{R}} 1\right)$$

Integration space

$$\forall_{i\in I}\forall_{\kappa\in\mathbb{F}(\mathbb{N}^+,I)}\left\{\left[\sum_{n\in\mathbb{N}^+}\int f_{\kappa(n)}\in\mathbb{R}\ \&\ \sum_{n\in\mathbb{N}^+}\int f_{\kappa(n)}<\int f_i\right]\Rightarrow\right.$$

$$\exists_{(\Phi,u)\,\in\,\left(\bigcap_{n\in\mathbb{N}^+}\lambda_0(\kappa(n))\right)\,\cap\,\lambda_0(i)}\left(\left(\sum_{n\in\mathbb{N}^+}^{\cap}f_{\kappa(n)}\right)(\Phi):=\sum_{n\in\mathbb{N}^+}f^{\mathbb{R}}_{\kappa(n)}(\Phi_n)\in\mathbb{R}\right.$$

$$\&$$

$$\left.\left.\sum_{n\in\mathbb{N}^+}f^{\mathbb{R}}_{\kappa(n)}(\Phi_n)<f^{\mathbb{R}}_i(u).\right)\right\}$$

$$\forall_{i\in I}\forall_{\alpha\in\mathbb{F}(\mathbb{N}^+,I)}\left(\forall_{n\in\mathbb{N}^+}\left(n\left(\frac{1}{n}f_i\ \wedge\ 1\right)=_{\mathfrak{F}(X)} f_{\alpha(n)}\right)\Rightarrow\right.$$

$$\left.\lim_{n\longrightarrow+\infty}\int f_{\alpha(n)}\in\mathbb{R}\ \ \&\ \lim_{n\longrightarrow+\infty}\int f_{\alpha(n)}=_{\mathbb{R}}\int f_i\right)$$

$$\forall_{i\in I}\forall_{\alpha\in\mathbb{F}(\mathbb{N}^+,I)}\left(\forall_{n\in\mathbb{N}^+}\left(\frac{1}{n}\big(n|f_i|\ \wedge\ 1\big)=_{\mathfrak{F}(X)} f_{\alpha(n)}\right)\Rightarrow\right.$$

$$\left.\lim_{n\longrightarrow+\infty}\int f_{\alpha(n)}\in\mathbb{R}\ \ \&\ \lim_{n\longrightarrow+\infty}\int f_{\alpha(n)}=_{\mathbb{R}}0\right)$$

Pre-integration space within BST

Let $(X, =_X, \neq_X)$ be an inhabited set, and let the set $(I, =_I)$ be equipped with operations $\cdot_a \colon I \rightsquigarrow I$, for every $a \in \mathbb{R}$, $+\colon I \times I \rightsquigarrow I$, $|.|\colon I \rightsquigarrow I$, and $\wedge_1 \colon I \rightsquigarrow I$, where

$$\cdot_a(i) := a \cdot i, \qquad +(i,j) := i + j, \qquad |.|(i) := |i|; \qquad i \in I, \ a \in \mathbb{R}.$$

Let also the operation $\wedge_a \colon I \rightsquigarrow I$, defined by the previous operations with the rule

$$\wedge_a := \cdot_a \circ \wedge_1 \circ \cdot_{a^{-1}}; \qquad a \in \mathbb{R} \ \& \ a > 0.$$

Let $\Lambda(X, \mathbb{R}) := (\lambda_0, \mathcal{E}^X, \lambda_1, \mathcal{P}^{\mathbb{R}}) \in \mathbf{Set}(I, X, \mathbb{R})$ i.e., $f_i =_{\mathfrak{F}(X)} f_j \Rightarrow i =_I j$, for every $i, j \in I$, and $f_i := \big(\lambda_0(i), \mathcal{E}_i^X, f_i^{\mathbb{R}}\big)$ is strongly extensional, for every $i \in I$. Let also a mapping

$$\int \colon I \to \mathbb{R}, \qquad i \mapsto \int i; \qquad i \in I,$$

such that the following conditions hold:

Pre-integration space

$$\forall_{i \in I} \forall_{a \in \mathbb{R}} \left(af_i =_{\mathfrak{F}(X)} f_{a \cdot i} \;\; \& \;\; \int a \cdot i =_{\mathbb{R}} a \int i \right)$$

$$\forall_{i,j \in I} \left(f_i + f_j =_{\mathfrak{F}(X)} f_{i+j} \;\; \& \;\; \int (i + j) =_{\mathbb{R}} \int i \; + \int j \right)$$

$$\forall_{i \in I} \left(|f_i| =_{\mathfrak{F}(X)} f_{|i|} \right)$$

$$\forall_{i \in I} \left(f_i \wedge 1 =_{\mathfrak{F}(X)} f_{\wedge_1(i)} \right)$$

$$\exists_{i \in I} \left(\int i =_{\mathbb{R}} 1 \right)$$

Pre-integration space

$$\forall_{i\in I}\forall_{\kappa\in\mathbb{F}(\mathbb{N}^+,I)}\Bigg\{\Bigg[\sum_{n\in\mathbb{N}^+}\int\kappa(n)\in\mathbb{R}\ \&\ \sum_{n\in\mathbb{N}^+}\int\kappa(n)<\int i\Bigg]\Rightarrow$$

$$\exists_{(\Phi,u)\in\left(\bigcap_{n\in\mathbb{N}^+}\lambda_0(\kappa(n))\right)\cap\lambda_0(i)}\Bigg(\Bigg(\sum_{n\in\mathbb{N}^+}^{\cap}f_{\kappa(n)}\Bigg)(\Phi):=\sum_{n\in\mathbb{N}^+}f^{\mathbb{R}}_{\kappa(n)}(\Phi_n)\in\mathbb{R}$$

$$\&\quad\sum_{n\in\mathbb{N}^+}f^{\mathbb{R}}_{\kappa(n)}(\Phi_n)<f^{\mathbb{R}}_i(u)\Bigg)\Bigg\}$$

Pre-integration space

$$\forall_{i \in I} \left(\lim_{n \longrightarrow +\infty} \int \wedge_n(i) \in \mathbb{R} \;\; \& \;\; \lim_{n \longrightarrow +\infty} \int \wedge_n(i) =_{\mathbb{R}} \int i \right)$$

$$\forall_{i \in I} \left(\lim_{n \longrightarrow +\infty} \int \wedge_{\frac{1}{n}}(|i|) \in \mathbb{R} \;\; \& \;\; \lim_{n \longrightarrow +\infty} \int \wedge_{\frac{1}{n}}(|i|) =_{\mathbb{R}} 0 \right)$$

A pre-integration space induces an integration space, if

$$\forall_{i \in I} \forall_{a \in \mathbb{R}^+} \left(f_i \;\; \wedge \;\; a =_{\mathfrak{F}(X)} f_j \Rightarrow \wedge_a(i) =_I j \right),$$

$$\int^* f_i := \int i; \qquad i \in I.$$

Locally compact metric space $(X, d, (K_n)_{n\in\mathbb{N}}, \boldsymbol{\kappa})$

Let (X, d) be an inhabited metric space with $x_0 \in X$, and let $(K_n)_{n\in\mathbb{N}}$ be a sequence of compact subsets of X. A modulus of local compactness for X is a function

$$\boldsymbol{\kappa}\colon \mathbb{N} \to \mathbb{N},$$

$$n \mapsto \boldsymbol{\kappa}(n),$$

such that $[d_{x_0} \leqslant n] \subseteq K_{\boldsymbol{\kappa}(n)}$, for every $n \in \mathbb{N}$.

If (Y, ρ) is an arbitrary metric space, $f\colon X \to Y$ is continuous, if it is uniformly continuous on $[d_{x_0} \leqslant n]$, for every $n \in \mathbb{N}$.

$\mathrm{Bic}(X, Y)$ the set of continuous functions from X to Y.

$\mathrm{Bic}(X) := \mathrm{Bic}(X, \mathbb{R})$.

Predicative definition of $C^{\mathrm{supp}}(X)$

If K is a compact subset of X and $f : X \to \mathbb{R}$, K is a support of f, if

$$\forall_{x \in X}\big(d(x, K) > 0 \Rightarrow f(x) = 0\big).$$

A function $f : X \to \mathbb{R}$ has compact support in $\big(X, d, (K_n)_{n\in\mathbb{N}}, \boldsymbol{\kappa}\big)$, if there is some $m \in \mathbb{N}$ such that K_m is a support of f.

$$C^{\mathrm{supp}}(X) := \big\{f \in \mathrm{Bic}(X) \mid f \text{ has compact support in } X\big\}.$$

The family of partial functions over $C^{\mathrm{supp}}(X)$

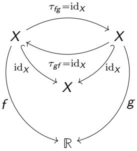
$\big(X, d, (K_n)_{n\in\mathbb{N}}, \boldsymbol{\kappa}\big)$ l.c.m.s., $\neq^d_X$, $I := C^{\mathrm{supp}}(X)$.

$$\tau_0 \colon I \rightsquigarrow \mathbb{V}_0, \quad \tau_0(f) := \big(X, \mathrm{id}_X, f\big); \quad f \in I,$$

$$\mathcal{E}^X \colon \bigcurlywedge_{f\in I} \mathbb{F}(X,X), \quad \mathcal{E}^X_f := \mathrm{id}_X; \quad f \in I,$$

$$\tau_1 \colon \bigcurlywedge_{(f,g)\in D(I)} \mathbb{F}(X,X), \quad \tau_1(f,g) := \tau_{fg}; \quad (f,g) \in D(I),$$

$$\mathcal{P}^{\mathbb{R}} \colon \bigcurlywedge_{f\in I} \mathbb{F}(X), \quad \mathcal{P}^{\mathbb{R}}_f := f; \quad f \in I,$$



$\mathrm{Supp}(X, \mathbb{R}) := \big(\tau_0, \mathcal{E}^X, \tau_1, \mathcal{P}^{\mathbb{R}}\big)$ is a set of strongly extensional real-valued, partial functions over I i.e., $\mathrm{Supp}(X,\mathbb{R}) \in \mathtt{Set}(I, X, \mathbb{R})$.

The pre-integration space of test functions of a l.c.m.s.

Let $\big(X, d, (K_n)_{n\in\mathbb{N}}, \boldsymbol{\kappa}\big)$ be a l.c.m.c., $\neq_X^d$ the canonical inequality on X, and let $I := C^{\mathrm{supp}}(X)$. Let the following operations on I:

(a) *If* $a \in \mathbb{R}$, *then* $\cdot_a : I \to I$ *is defined by* $f \mapsto af$.

(ii) $+: I \times I \to I$ *is the addition of functions on* I.

(iii) $|.|: I \to I$ *is defined by* $f \mapsto |f|$.

(iv) $\wedge_1 : I \to I$ *is defined by* $f \mapsto f \wedge \overline{1}^X$.

(v) *If* $a > 0 \in \mathbb{R}$, *then* $\wedge_a : I \to I$ *is defined as the composition* $\wedge_a := \cdot_a \circ \wedge_1 \circ \cdot_{a^{-1}}$.

Let $\mathrm{Supp}(X, \mathbb{R}) := \big(\tau_0, \mathcal{E}^X, \tau_1, \mathcal{P}^{\mathbb{R}}\big)$ *and* $\mu : I \to \mathbb{R}$ *a positive measure on* X *i.e.,* μ *is a nonzero* (*i.e., there is* $f \in I$ *with* $\mu(f) > 0$), *positive, linear map on* I, *and let*

$$\int -d\mu : I \to \mathbb{R}, \qquad f \mapsto \int f d\mu := \mu(f); \qquad f \in I.$$

Then $\big(X, I, \int -d\mu\big)$ *is a pre-integration space.*

Using set-indexed families of complemented subsets of some $(X, =_X, \neq_X)$, the notion of pre-measure space is defined similarly.

The pre-integration space of the simple functions of a pre-measure space is defined in [28] and [19].

There a predicative approach to L^1 is presented through the canonically integrable functions, it is shown that L^1 is a pre-integration space that completes the original pre-integration space L, in a certain sense.

The resulting theory is predicative, and (countable) choice is avoided.

The integration theory of locally compact metric spaces is generalised to the integration theory of Bishop spaces in [17].

We fix a pre-integration space $\big(X, I, \Lambda(X,\mathbb{R}), \textstyle\int\big)$.

$$I_1 := \left\{\alpha \in \mathbb{F}(\mathbb{N}, I) \mid \sum_{n=1}^{\infty} \int |\alpha_n| \in \mathbb{R}\right\}$$

$$\alpha =_{I_1} \beta :\Leftrightarrow \Big(F_\alpha, e_{F_\alpha}, \sum_n f_{\alpha_n}\Big) =_{\mathfrak{F}^{\mathrm{se}}(X)} \Big(F_\beta, e_{F_\beta}, \sum_n f_{\beta_n}\Big)$$

$$F_\alpha := \left\{x \in \bigcap_n \lambda_0\big(\alpha(n)\big) \mid \sum_n |f_{\alpha(n)}(x)| \in \mathbb{R}\right\}$$

The canonically integrable functions is the I_1-family of partial functions $\mathbf{\Lambda}_1 := \big(\nu_0, \mathcal{H}^X, \nu_1, \mathcal{R}\big)$

- $\nu_0 : I_1 \rightsquigarrow \mathbb{V}_0$, $\nu_0(\alpha) := F_\alpha$.
- $\mathcal{H}^X : \curlywedge_{\alpha \in I_1} \mathbb{F}(\nu_0(\alpha), X)$, $\alpha \in I_1 \mapsto e_\alpha := e_{F_\alpha}$.
- $\nu_1 : \curlywedge_{(\alpha,\beta) \in D(I_1)} \mathbb{F}(\nu_0(\alpha), \nu_0(\beta))$,

  $(\alpha, \beta) \in D(I_1) \mapsto \nu_{\alpha\beta} : \nu_0(\alpha) \to \nu_0(\beta)$ s.t. $\nu_{\alpha\alpha} := \mathrm{id}_{\nu_0(\alpha)}$ and $(\nu_{\alpha\beta}, \nu_{\beta\alpha})$ witnesses the equality $\alpha =_{I_1} \beta$
- $\mathcal{R} : \curlywedge_{\alpha \in I_1} \mathbb{F}(\nu_0(\alpha), \mathbb{R})$, $\alpha \in I_1 \mapsto g_\alpha : \nu_0(\alpha) \to \mathbb{R}$
  $g_\alpha(x) := \sum_{n=1}^{\infty} g_{\alpha_n)}(x)$ for each $x \in \nu_0(\alpha)$.

$$i \mapsto \big(i, 0 \cdot i, 0 \cdot i, \ldots\big)$$

$$\_ +_1 \_ : I_1 \times I_1 \to I_1$$

$$\alpha +_1 \beta := \big(\alpha(1), \beta(1), \alpha(2), \beta(2), \ldots \big),$$

$$\_ \cdot_1 \_ : \mathbb{R} \times I_1 \to I_1$$

$$a \cdot_1 \alpha := \big(a{\cdot}\alpha(1), a{\cdot}\alpha(2), \ldots \big)$$

$$|\_|_1 : I_1 \to I_1$$

$$|\alpha|_1 := \big(|\alpha(1)|, \alpha(1), (-1){\cdot}\alpha(1), |\alpha(1)+\alpha(2)|-|\alpha(1)|, \alpha(2), (-1){\cdot}\alpha(2), |\alpha(1)+\alpha(2)+\alpha(3)|-$$

$$\int^1 : I_1 \rightsquigarrow \mathbb{R}$$

$$\int^1 \alpha := \sum_n \int \alpha(n)$$

is a function and

$$\int^1 i = \int i$$

Lebesgue's series theorem

Let $\Gamma \in \mathbb{F}(\mathbb{N}, I_1)$ s.t. $\sum_n \int |\Gamma(n)| \in \mathbb{R}$. Then there is an $\alpha \in I_1$ s.t.

$$\nu_0(\alpha) \subseteq \left\{ x \in \bigcap_{n=1}^{\infty} \nu_0\big(\Gamma(n)\big) \;\Big|\; \sum_{n=1}^{\infty} |g_{\Gamma(n)}(x)| \in \mathbb{R} \right\}$$

$$\& \; \forall_{x \in \nu_0(\alpha)} \left(g_\alpha(x) = \sum_{n=1}^{\infty} g_{\Gamma(n)}(x) \right)$$

Furthermore, for every $\alpha \in I_1$ as above we have

$$\lim_{N \to \infty} \int^1 \left| \alpha - \sum_{n=1}^{N} \Gamma(n) \right| = 0$$

Theorem
$\left(X, I_1, \mathbf{\Lambda}_1, \int^1\right)$ *is a pre-integration space.*

$$i =_{\int} j \;:\Leftrightarrow\; \int |i - j| = 0$$

is an equivalence relation on I and the assignment routine $\int : (I, =_{\int}) \rightsquigarrow \mathbb{R}$ given by $i \mapsto \int i$ is a function. Moreover, the functions $\cdot$ and $+$ make $(I, =_{\int})$ into an $\mathbb{R}$-vector space with neutral element $0 \cdot p$, where $p \in I$ is s.t. $\int p = 1$.

The function $||.||_1 : I \to \mathbb{R}$

$$||i||_1 := \int |i|$$

is a norm. With similar definitions of equality and norm on I_1:

I is dense in the complete normed space I_1.

$\boldsymbol{A}$ is integrable, if $\chi_{\boldsymbol{A}} \in L^1$, and the induced measure of $\boldsymbol{A}$ is given by

$$\mu(A) := \int^1 \chi_{\boldsymbol{A}}$$

It is a (pre-)measure space (constructive Stone theorem)

Theory of profiles to get many examples of integrable sets

Measurable functions are "approximated" by integrable ones:
$f : X \rightharpoonup \mathbb{R}$ with $\mathrm{dom}(f)$ full is measurable, if for every integrable $\boldsymbol{A}$ and $\epsilon > 0$ there are integrable $\boldsymbol{B}$ and $g \in L^1$ s.t.

$$B^1 \subseteq A^1 \;\;\&\;\; \mu(\boldsymbol{A} - \boldsymbol{B}) < \epsilon \;\;\&\;\; |f - g| < \epsilon \text{ on } B^1$$

$\boldsymbol{A}$ is measurable, if $\chi_{\boldsymbol{A}}$ is measurable

- E. Bishop: *Foundations of Constructive Analysis*, McGraw-Hill, 1967.
- S. Bhat, A. Agarwal, R. Vuduc: A type theory for probability density functions, POPL'12, ACM, 2012, 545–556.
- E. Bishop, H. Cheng: *Constructive Measure Theory*, Mem. Amer. Math. Soc. 116, 1972.
- E. Bishop, D. S. Bridges: *Constructive Analysis*, Grundlehren der math. Wissenschaften 279, Springer-Verlag, Heidelberg-Berlin-New York, 1985.
- D. Bridges, O. Demuth: On the Lebesgue measurability of continuous functions in constructive analysis, Bulletin of the American Mathematical Society, Vol. 24, No. 2, 1991, 259–276.
- C. Carathéodory: *Mass und Integral und ihre algebraisierung*, Springer Basel AG, 1956.
- T. Coquand, E. Palmgren: Metric Boolean algebras and constructive measure theory, Arch. Math. Logic 41, 2002, 687–704.

P. J. Daniell: A general form of integral, Annals of mathematics, Second Series, 19 (4), 1918, 279–294.

A. Edalat: A computable approach to measure and integration theory, Information and Computation 207, 2009, 642–659.

F. Faissole, B. Spitters: Synthetic topology in Homotopy Type Theory for probabilistic programming, preprint, 2016.

A. Heyting: *Intuitionism, An introduction*, North-Holland, 1956.

H. Ishihara: A constructive integration theory: a topological approach, preprint, 2020.

A. N. Kolmogoroff: Algèbres de Boole métrique complètes, VI Zjazd Mathematyków Polskich, Warwaw, 1948, 21–30.

I. Petrakis: Dependent Sums and Dependent Products in Bishop's Set Theory, in P. Dybjer et. al. (Eds) TYPES 2018, LIPIcs, Vol. 130, Article No. 3, 2019.

I. Petrakis: *Families of Sets in Bishop Set Theory*, Habilitationsschrift, LMU, Munich, 2020.

- I. Petrakis: Chu representations of categories related to constructive mathematics, arXiv:2106.01878v1, 2021.
- I. Petrakis: Integration theory of Bishop spaces, in preparation, 2021.
- I. Petrakis: Constructive and predicative integration theory of locally compact metric spaces, in preparation, 2021.
- I. Petrakis, M. Zeuner: Pre-measure spaces and pre-integration spaces in predicative Bishop-Cheng measure theory, in preparation, 2021.
- N. Šanin: *Constructive Real Numbers and Function Spaces*, Translations of Mathematical Monographs, vol. 21, AMS, Providence Rhode Island, 1968.
- I. Segal: Abstract probability spaces and a theorem of Kolmogoroff, Amer. J. Math. 76, 1954, 721–732.
- I. Segal: Algebraic integration theory, Bull. Amer. Math. Soc. 71, 1965, 419–489.

- M. Shulman: Linear Logic for Constructive Mathematics, arXiv:1805.07518v1, 2018.
- B. Spitters: Constructive algebraic integration theory. Ann. Pure Appl. Logic 137(1-3), 2006, 380–390.
- M. H. Stone: Notes on integration: II, Proc. Nat. Acad. Sci. U.S.A. 34, 1948, 447-455.
- S. J. Taylor: *Introduction to Measure and Integration*, Cambridge University Press, 1973.
- A. Weil: Calcul des probabilités, méthode axiomatique, intégration, Revue Sci. (Rev. Rose Illus.) 78, 1940, 201–208.
- M. Zeuner: *Families of Sets in Constructive Measure Theory*, Master's Thesis, LMU, 2019.