Computable Multifunctions on Effectively Hausdorff Spaces

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Survey

- The current notion of computable Hausdorffness
- A new notion of effective Hausdorffness
- A characterisation of computable multifunctions
- Strongly computable multifunctions

The current notion of computable Hausdorffness

Current Definition

X is *computably Hausdorff*, if inequality on X is semi-decidable.

Example

Any computable metric space.

Characterisation (A. Pauly 2012)

Let X be a computably admissible represented T_1-space. TFAE:

- X is computably Hausdorff.
- The diagonal $\{(x, x) \mid x \in X\}$ is co-c.e. closed.
- The embedding $X \hookrightarrow \mathcal{A}(X)$, $x \mapsto \{x\}$ is computable.
- The inclusion $\mathcal{K}(X) \hookrightarrow \mathcal{A}(X)$ is well-defined and computable.
- $\cap \colon \mathcal{K}(X) \times \mathcal{K}(X) \to \mathcal{K}(X)$ is well-defined and computable.

Main Disadvantage

Computable Hausdorffness $\not\Rightarrow$ topological Hausdorffness.

Counterexample

Let $\omega\mathsf{M}$ be the one-point compactification of a computable metric space M that is *not* locally compact.

- $\omega\mathsf{M}$ is computably Hausdorff,
- but not topologically Hausdorff.

One-point compactification of M:

- Underlying set: $\omega\mathsf{M} := \mathsf{M} \uplus \{\omega\}$
- Topology: $\mathrm{O}(\mathsf{M}) \cup \{\omega\mathsf{M} \setminus K \mid K$ compact in $\mathsf{M}\}$
- $\omega\mathsf{M}$ has a canonical representation $\delta_{\omega\mathsf{M}}$ derived from δ_{M}.

Disadvantage II

Computable Hausdorffness does not admit effectivisation of some classical theorems.

Example

- *Classical Theorem:* Any compact Hausdorff space is regular.
- *But:* A computably compact, computably Hausdorff space need *not* be computably regular.
- *Counterexample:* The one-point compactification $\omega\mathrm{M}$ of a computable metric space M that is *not* locally compact.

Remember

- X is *computably compact*, if, for U open, '$U = X$?' is semi-decidable.
- X is *computably regular*, if, given $x \in U \in \mathcal{O}(X)$, one can computably select an open set V and a closed set A with $x \in V \subseteq A \subseteq U$.

The new notion of effective Hausdorffness

Basic Idea

Proposition

Let X be a Hausdorff QCB-space.

- X has a subtopology $\tau \subseteq \mathcal{O}(X)$ that is countably based and Hausdorff.
- Any such subtopology τ satisfies:
  - $\tau|_K = \mathcal{O}(X)|_K$ for any compact subspace $K \in \mathcal{K}(X)$.
  - $(x_n)_n$ converges to x_∞ in X iff
    (a) $(x_n)_n$ converges to x_∞ w.r.t. τ &
    (b) $(x_n)_n$ is contained in some $K \in \mathcal{K}(X)$.

Remember

- QCB-space $=$ a **q**uotient of a **c**ountably **b**ased topological space
- QCB $=$ class of top. spaces that can be handled by TTE

Definition

A *computable witness of Hausdorffness* for X is a sequence $(\mathrm{u}_i, \mathrm{v}_i)_i$ in $\mathcal{O}(\mathsf{X}) \times \mathcal{O}(\mathsf{X})$ such that:

- $\mathrm{u}_i \cap \mathrm{v}_i = \emptyset$ for all $i \in \mathbb{N}$.
- For all $x \neq y$, there is some j such that $x \in \mathrm{u}_j$, $y \in \mathrm{v}_j$.
- $(u_i)_i$, $(v_i)_i$ are computable sequences (w.r.t. the canonical positive representation θ_+ of $\mathcal{O}(\mathsf{X})$).

Definition

We call a rep. space X an *effectively Hausdorff QCB-space*, if

- it has a computable witness of Hausdorffness &
- its representation δ_{X} is computably admissible.

Example (Effectively Hausdorff QCB-space)
Any computable metric space.

Observation
Any effectively Hausdorff space is

- topologically Hausdorff,
- computably Hausdorff.

Proposition [A. Pauly 2021]
Topologically Hausdorff $\wedge$ computably Hausdorff $\not\Longrightarrow$ effectively Hausdorff.

Theorem

Let X and Y be effectively Hausdorff QCB-spaces.

Then the following spaces are effectively Hausdorff:

- $X \times Y$
- $X \oplus Y$
- any QCB-subspace of X
- Y^Z, whenever Z has a computable dense sequence

Classical Theorems

- Any compact Hausdorff space is regular.
- Any separable compact Hausdorff space is metrisable.

Theorem

- Any computably compact, effectively Hausdorff space X is computably regular.
- If additionally X has a computable dense sequence $(\alpha_k)_k$, then X has a metric d such that
  - (X, d, α) is a computable metric space &
  - its Cauchy representation is computably equivalent to δ_X.

A characterisation of computable multifunctions

Recap

- A *multifunction* *(or computational problem)* F is a relation between represented spaces X, Y, written as $F\colon \mathsf{X} \rightrightarrows \mathsf{Y}$.
- X is the *input space*, Y is the *output space* of F.
- Notation: $F[x] := \left\{ y \in \mathsf{Y} \,\middle|\, (x, y) \in F \right\}$.

Remark

We will assume every multifunction F to be *total*, i.e. $F[x] \neq \emptyset$ for all $x \in \mathsf{X}$.

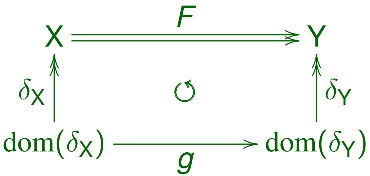
Recap

Let $F\colon \mathsf{X} \rightrightarrows \mathsf{Y}$ be a total multifunction.

- F is called *computable*, if there is a computable *realizer* $g\colon \mathbb{N}^{\mathbb{N}} \dashrightarrow \mathbb{N}^{\mathbb{N}}$ satisfying

$$\delta_{\mathsf{Y}} g(p) \in F[\delta_{\mathsf{X}}(p)] \qquad \text{for all } p \in \mathrm{dom}(\delta_{\mathsf{X}}).$$

- Diagram:



Powerspaces of compact subsets

Definition [CCA 2020]

Let Y be a represented QCB_2-space.

- $\mathrm{K}(\mathsf{Y}) := \{$all non-empty compact subsets of $\mathsf{Y}\}$.
- Using
  - κ_+ (= the canonical positive representation for $\mathrm{K}(\mathsf{Y})$ in TTE)
  - κ_- (= the canonical negative representation for $\mathrm{K}(\mathsf{Y})$ in TTE)

  we define a representation $\kappa_{+\mathrm{b}}$ for $\mathrm{K}(\mathsf{Y})$ by

$$\kappa_{+\mathrm{b}}\langle p, b\rangle = K \quad \text{iff} \quad \kappa_+(p) = K \;\&\; K \subseteq \kappa_-(b).$$

- Set:
  - $\mathcal{K}_{+\mathrm{b}}(\mathsf{Y}) := \big(\mathrm{K}(\mathsf{Y}), \kappa_{+\mathrm{b}}\big)$
  - $\mathcal{K}_{+}(\mathsf{Y}) := \big(\mathrm{K}(\mathsf{Y}), \kappa_{+}\big)$
  - $\mathcal{K}_{-}(\mathsf{Y}) := \big(\mathrm{K}(\mathsf{Y}), \kappa_{-}\big)$

Characterisation Theorem

Let X be a computable metric space and Y be effectively Hausdorff. Let $F\colon X \rightrightarrows Y$ be a total multifunction. TFAE:

(a) F is computable.

(b) There is a computable function $h\colon X \to \mathcal{K}_{+\mathrm{b}}(Y)$ such that

$$\emptyset \neq h(x) \subseteq F[x] \quad \text{for all } x \in X.$$

(c) There are computable functions $h_+\colon X \to \mathcal{K}_+(Y)$ and $h_\mathrm{b}\colon X \to \mathcal{K}_-(Y)$ such that

$$\emptyset \neq h_+(x) \subseteq F[x] \cap h_\mathrm{b}(x) \quad \text{for all } x \in X.$$

Remark

- (b) $\not\Longrightarrow$ (a) for computably Hausdorff spaces Y.
- (a) $\not\Longrightarrow$ (b) for non-metrisable spaces X.
- (b) $\Longrightarrow$ (a) needs only effective Hausdorffness of Y.
- (a) $\Longrightarrow$ (b) needs only computable metrisability of X.

"(b) $\Longrightarrow$ (a)" *is based on:*

Theorem

Let Y be effectively Hausdorff. Then:

- Compact choice for Y is computable w.r.t. $\kappa_{+\mathrm{b}}$,
- *i.e.:* there is a computable selector $S\colon \mathrm{dom}(\kappa_{+\mathrm{b}}) \to \mathsf{Y}$ such that $S(p) \in \kappa_{+\mathrm{b}}(p)$.

Remark

- This is not true for computably Hausdorff spaces Y.
- For $\mathsf{Y} \in \mathsf{QCB}_2 \setminus \omega\mathsf{Top}$, compact choice is incomputable w.r.t. κ_{+}.
- Compact choice is computable w.r.t. κ_{+} for:
  - [V. Brattka & P. Hertling 1994]
    any computable metric space
  - [M. de Brecht & A. Pauly & Sch. 2019]
    any computably Hausdorff, computable quasi-Polish space

Strongly computable multifunctions

Goal

A computability notion for multifunctions which

- is based on computable set-valued functions
- works for non-metrisable input spaces X
- entails the usual notion

Recall

The Characterisation Theorem only works for input spaces X that are computably metrisable.

Recall

$F\colon \mathsf{X} \rightrightarrows \mathsf{Y}$ is computable iff there are computable h_+, h_b into $\mathrm{K}(\mathsf{Y})$ such that $\emptyset \neq h_+(x) \subseteq F[x] \cap h_b(x)$.

Definition

We call $F\colon \mathsf{X} \rightrightarrows \mathsf{Y}$ *strongly computable*, if there are computable functions $h_+\colon \mathsf{X} \to \mathcal{K}_+(\mathsf{Y})$ and $h_-\colon \mathsf{X} \to \mathcal{K}_-(\mathsf{Y})$ such that

$$\emptyset \neq h_+(x) \subseteq F[x] \subseteq h_-(x) \quad \text{for all } x \in \mathsf{X}.$$

Alternative Definition

Call $F\colon \mathsf{X} \rightrightarrows \mathsf{Y}$ *strongly computable*, if F has compact images and is computable viewed as a map from X to $\mathcal{K}_{+\mathrm{b}}(\mathsf{Y})$.

Example (Strongly computable multifunctions)

- The finite precision test $(x <_\varepsilon y) \colon \mathbb{R} \times \mathbb{R}_{>0} \times \mathbb{R} \rightrightarrows \mathrm{Bool}$,

$$(x <_\varepsilon y) := \begin{cases} \{\mathtt{true}\} & \text{if } x \le y - \varepsilon \\ \{\mathtt{true}, \mathtt{false}\} & \text{if } y - \varepsilon < x < y \\ \{\mathtt{false}\} & \text{if } x \ge y \end{cases}$$

- Zero-finding $Z_3 \colon \mathbb{R}^3 \rightrightarrows \mathbb{R}$ of polynomials of degree 3,

$$Z_3[a, b, c] := \left\{ x \in \mathbb{R} \,\middle|\, x^3 + ax^2 + bx + c = 0 \right\}$$

- The inverse of the signed-digit representation.

Theorem

Let X be a represented space and Y be effectively Hausdorff. Then any strongly computable total multifunction $F\colon \mathsf{X} \rightrightarrows \mathsf{Y}$ is computable.

Proposition

Let X be a comp. metric space and Y be effectively Hausdorff. Then for any computable $G\colon \mathsf{X} \rightrightarrows \mathsf{Y}$ there exists a strongly computable $F\colon \mathsf{X} \rightrightarrows \mathsf{Y}$ such that, for all $x \in \mathsf{X}$,

- $\emptyset \neq F[x] \subseteq G[x]$
- $F[x]$ is compact.

Definition (The composition $\diamond$)

For total multifunctions $F\colon \mathrm{X} \rightrightarrows \mathrm{Y}$, $G\colon \mathrm{Y} \rightrightarrows \mathrm{Z}$ define

$$G \diamond F[x] := \mathrm{Cls}\big(G[F[x]]\big) = \mathrm{Cls}\bigcup\big\{G[y] : y \in F[x]\big\}$$

for all $x \in \mathrm{X}$.

Proposition

The composition $\diamond$ preserves strong computability for total multifunctions between effectively Hausdorff spaces.

Proposition

- Strongly computable multifunctions with compact images between effectively Hausdorff spaces form a category.
- This category EffHausMulti has equalisers and finite coproducts, but only weak products.

Summary

- The new notion of effective Hausdorffness entails
  - the previous notion,
  - topological Hausdorffness,
  - effective versions of some classical theorems from topology.
- Computable multifunctions from computable metric spaces to effective Hausdorff spaces can be characterised via compact-set-valued functions.
- Strong computability of multifunctions
  - is defined via computability of set-valued functions,
  - works for all effectively Hausdorff spaces.
- Open problem:

  Find a characterisation of computable multifunctions on input spaces that are not computable metric spaces.

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