Descriptive complexity of topological invariants

Djamel Eddine Amir and Mathieu Hoyrup
Université de Lorraine, CNRS, Inria, LORIA, F-54000 Nancy, France

We study the descriptive complexity of topological invariants of compact Polish spaces. Many usual invariants are known to have non-Borel complexity (see [Bec92, DS20] for instance), and we are interested in the expressiveness of low-complexity invariants. We fully characterize the Π^0_1 invariants, which are all about connectedness, and obtain several results about Σ^0_2 invariants. We identify a natural Σ^0_2 invariant, which expresses the existence of a non-trivial mapping from the space to the n-sphere.

We are mostly interested in Σ^0_2 invariants because they can be used to prove that certain sets are computable as soon as they are Π^0_1. Miller [Mil02] proved that if a Π^0_1-subset of a Euclidean space is homeomorphic to a sphere then it is computable. Iljazovic [IS18] generalized this result to closed manifolds embedded in the Hilbert cube. We show how such results can be obtained by observing that the set is minimal satisfying some Σ^0_2 invariant. In particular, the Σ^0_2 invariant mentioned above implies the result for spheres and closed manifolds. New results in this direction can be easily obtained using this minimality argument.

References

[Bec92] Howard Becker. Descriptive set theoretic phenomena in analysis and topology. In Haim Judah, Winfried Just, and Hugh Woodin, editors, *Set Theory of the Continuum*, pages 1–25, New York, NY, 1992. Springer US.

[DS20] Gabriel Debs and Jean Saint Raymond. The descriptive complexity of connectedness in Polish spaces. *Fundamenta Mathematicae*, 249(3):261–286, 2020.

[IS18] Zvonko Iljazović and Igor Sušić. Semicomputable manifolds in computable topological spaces. *J. Complex.*, 45:83–114, 2018.

[Mil02] Joseph S. Miller. Effectiveness for embedded spheres and balls. *Electronic Notes in Theoretical Computer Science*, 66(1):127 – 138, 2002. CCA 2002, Computability and Complexity in Analysis (ICALP 2002 Satellite Workshop).