Poisson generic real numbers

Verónica Becher
Universidad de Buenos Aires

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The present talk is based on joint work with Nicolás Alvarez and Martín Mereb.

Years ago Zeev Rudnick defined Poisson genericity as a property of real numbers, by counting the number of occurrences of long words in the initial segments of the fractional expansion of a real number in a given integer base[1].

Let b be an integer greater than or equal to 2 and let $\Omega = \{0, \ldots, b-1\}$. The initial segments of the base-b expansion of a random real number x can be seen as follows: for each positive integer k, the first N symbols of this expansion are N almost independent events of words of length k each one with equal probability $p = b^{-k}$ –the almost independence is because the random events at two positions at distance less than k is b^{-2k}, as if they were independent—. The expected proportion of the b^k many words that occur exactly i times, for $i = 0, 1, \ldots$, is

$$\chi(i) = \binom{N}{j} p^i (1-p)^{N-i}.$$

When N/b^k is a fixed constant λ, $\chi(i)$ converges in probability to $e^{-\lambda}\lambda^i/i!$, hence, to the Poisson law with parameter λ, see [2, Example III.10 on the allocation of balls in bins].

For a positive real number λ, a non-negative integer i, a positive integer k and a real number x, let

$$Z^{\lambda}_{i,k}(x) = \frac{1}{b^k} \#\{\omega \in \Omega^k : \omega \text{ occurs } i \text{ times in the first } \lfloor \lambda b^k \rfloor \text{ symbols in the base-}b \text{ expansion of } x\}$$

For example for $b = 2$, $\lambda = 1$, $k = 3$, $b^k = \lfloor \lambda b^k \rfloor = 8$, and $x = 10000100\ldots$ we have:
for $i = 0$, $Z^{\lambda}_{i,k}(x) = 4/8$ (witnesses $011, 101, 110, 111$)
for $i = 1$, $Z^{\lambda}_{i,k}(x) = 2/8$ (witnesses $001, 010$)
for $i = 2$, $Z^{\lambda}_{i,k}(x) = 2/8$ (witnesses $100, 000$)
for $i \geq 3$, $Z^{\lambda}_{i,k}(x) = 0$.

Definition. *Let b be a positive integer greater than or equal to 2 and let λ be a positive real number. A real number x is λ-Poisson generic for base b if for all non-negative integers i,*

$$\lim_{k\to\infty} Z^{\lambda}_{i,k}(x) = e^{-\lambda}\frac{\lambda^i}{i!}.$$

A real number is Poisson generic for base b if it is λ-Poisson generic for base b for all positive real numbers λ.

The following metric result holds:

Theorem 1 (Peres and Weiss [5], see also [6, Theorem 1])**.** *Almost all real numbers in the unit interval, with respect to Lebesgue measure, are Poisson generic for every integer base.*

[1]Rudnick was motivated by his result in [3] that in lacunary sequences the number of elements in a random interval of the size of the mean spacing follows the Poisson law.

This result follows from the theorem proved by Yuval Peres and Benjamin Weiss [5] where they showed that a definition stronger than Poisson genericity (they consider all sets of positions definable from Borel sets instead of just initial intervals) yields a Poisson process in the real line.

Peres and Weiss also proved [4, 5] that Poisson genericity implies Borel normality and that the two notions are not equivalent, witnessed by the fact that Champernowne's sequence is not Poisson generic. We related Poisson genericity with the theory of computability:

Theorem 2 ([6, Theorem 2]). *For each integer base $b \geq 2$, there are countably many Poisson generic real numbers for base b in every Turing degree.*

Theorem 3 ([6, Theorem 3]). *All Schnorr random real numbers are Poisson generic for every integer base.*

We gathered statistics and we arrived to this:

Conjecture ([6]). *For every integer base b greater than or equal to 2, the real number whose base b-expansion sequence is the concatenation of the Fibonacci numbers written in base b is 1-Poisson generic for base b.*

Using some combinatorics on words we recently obtained together with Gabriel Sac Himelfarb:

Theorem 4 ([1, Theorem 1]). *For every base $b \geq 3$ and for every positive real number λ there is an explicit construction in linear time of a λ-Poisson generic number for base b.*

We still do not know how to make a construction for the case $b = 2$.

References

[1] Verónica Becher and Gabriel Sac Himelfarb. A construction of a λ-poisson generic sequence. *submitted*, 2022.

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[3] Zeev Rudnick and Aledandru Zaharescu. The distribution of spacings between fractional parts of lacunary sequences. *Forum Mathematicum*, 14(5):691–712, 2002.

[4] Benjamin Weiss. Random-like behavior in deterministic systems, 16 June 2010. Institute for Advanced Study Princeton University USA. `https://www.youtube.com/watch?v=8ab7591de68t=1567s`.

[5] Benjamin Weiss. Poisson generic points, 23-27 November 2020. Conference Diophantine Problems, Determinism and Randomness - CIRM- organized by Joël Rivat and Robert Tichy. `https://library.cirm-math.fr/Record.htm?idlist=13&record=19287810124910050929`.

[6] Nicolás Álvarez, Verónica Becher, and Martín Mereb. Poisson generic real numbers. *submitted*, 2022.