Embeddability of graphs and Weihrauch degrees

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An undirected graph $G_0 = (V_0, E_0)$ is a subgraph of $G = (V, E)$ if and only if $V_0 \subseteq V$ and $E_0 \subseteq E$. If $E_0 = E \cap (V_0 \times V_0)$ then G_0 is an induced subgraph of G. We first study the Weihrauch degree of decision problems IS_G parametrized with a countable graph G. An input for IS_G is a countable graph H and $\mathsf{IS}_G(H) = 1$ iff H has an induced subgraph isomorphic to G. A similar definition holds for S_G replacing the induced subgraph relation with the subgraph one. These principles were introduced and studied in [1] with different names, namely S_G (for IS_G) and SE_G (for S_G) respectively. The authors proved that for any non-empty computable graph G, $\mathrm{LPO} \le_{\mathrm{W}} \mathsf{IS}_G \le_{\mathrm{W}} \mathsf{WF}$ where LPO is the problem of deciding if a sequence in $2^{\mathbb{N}}$ has not only zeros and WF is the problem of deciding whether a countably branching tree is well-founded. Moreover, they showed that for finite G, $\mathrm{LPO} \equiv_{\mathrm{W}} \mathsf{IS}_G$, and for the infinite ray graph $R = (\mathbb{N}, \{(i, i+1) : i \in \mathbb{N}\})$, $\mathsf{WF} \equiv_{\mathrm{W}} \mathsf{IS}_R$. They left open whether there exists a (computable) graph G such that $\mathrm{LPO} <_{\mathrm{W}} \mathsf{IS}_G <_{\mathrm{W}} \mathsf{WF}$. We answer the question negatively, proving that for any infinite computable graph G, $\mathsf{IS}_G \equiv_{\mathrm{W}} \mathsf{WF}$. The situation is quite different when we consider the problem S_G. From [1] we know that there exists a computable graph G such that $\mathsf{S}_G \equiv_{\mathrm{W}} \mathrm{LPO}'$: here we show that for any $n \in \mathbb{N}$, there exists an infinite computable graph G such that $\mathrm{LPO}^{(n)} \equiv_{\mathrm{W}} \mathsf{S}_G$[1]

We then study the Weihrauch degree of the multi-valued functions FindIS_G and FindS_G, parametrized with countable graph G. The first takes in input a graph H such that H has an induced subgraph isomorphic to G and outputs an isomorphic copy of G in H, the second is defined similarly, replacing the induced subgraph relation with the subgraph one. While for finite graphs both problems are computable, we show that for any infinite computable graph G, $\mathsf{FindIS}_G \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$ (even if for some G the reduction is up to some oracle). The problem FindS_G behaves quite differently and we study for which graphs G we have that $\mathsf{FindS}_G \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$. We finally conclude with some remarks on the "opposite" problem of IS_G, namely IS^G. An input for IS^G is a countable graph H where $\mathsf{IS}^G(H) = 1$ iff G has an induced subgraph isomorphic to H. We (partially) solve a question that was left open in [1], proving that there exists a computable graph G such that $\mathrm{LPO} <_{\mathrm{W}} \mathsf{IS}^G$.

References

[1] Zach BeMent, Jeffry L. Hirst & Asuka Wallace (2021): *Reverse mathematics and Weihrauch analysis motivated by finite complexity theory.* *Computability* 10(4), pp. 343–354, doi:10.3233/COM-210310. Available at https://doi.org/10.3233/COM-210310. ArXiv 2105.01719.

[1] LPO' denotes the jump of LPO and $\mathrm{LPO}^{(n)}$ the n-fold jump of LPO. Informally, for a multivalued function f, f' is the same as f except for the fact that it receives in input a name converging to a name for an input of f.