EXTRACTION RATE OF RANDOM FUNCTIONALS

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In this paper, we extend previous work by the authors on a notion of extraction rate of Turing functionals that translate between different notions of randomness with respect to different underlying probability measures. Here, instead of focusing on computably continuous functionals, we consider the extraction rate of the *random continuous functionals* introduced by Barmpalias, Brodhead, Cenzer, et al in [BBC+08], and studied further by Cenzer and Porter in [CP15] and by Cenzer and Rojas in [CR18]. These are the functionals whose output bits are determined by rolling 3-sided dice–corresponding to outputs of 0, 1, or a blank, which represents a delay in output. We show that each random continuous functional has average extraction rate equal to the probability of producing one bit of output by die-toss. Further, we show that for every random continuous functional F, any input that is Martin-Löf random relative to F produces outputs per every bit of input at rate equal to the average extraction rate of F.

For a total continuous functional $F : 2^\omega \to 2^\omega$, we may define F via a function $f : 2^{<\omega} \to 2^{<\omega}$, which we refer to as a *representation* of F, satisfying the conditions that (i) if $\sigma \preceq \tau$, then $f(\sigma) \preceq f(\tau)$, and (ii) for all $X \in 2^\omega$, $\lim_{n\to\infty} |f(X{\upharpoonright}n)| = \infty$. We then have $F(X) = \bigcup_n f(X{\upharpoonright}n)$.

Partial continuous functionals $F :\subseteq 2^\omega \to 2^\omega$ are given by those $f : 2^{<\omega} \to 2^{<\omega}$ which only satisfy condition (i). In this case $F(X) = \bigcup_n f(X{\upharpoonright}n)$ may be only a finite string. A *Turing functional* is a partial continuous functional F with partial computable representation f, and F is called a *tt-functional* if f is total computable. We will use the term "continuous functionals" to refer to partial continuous functionals unless stated otherwise.

Recall that a measure μ on 2^ω is *computable* if there is a computable function $\phi : 2^{<\omega} \times \omega \to \mathbb{Q}_2$ such that $|\mu(\llbracket\sigma\rrbracket) - \phi(\sigma, i)| \leq 2^{-i}$, where $\llbracket\sigma\rrbracket$ is the set $\{X \in 2^\omega : \sigma \prec X\}$. Hereafter, we will write $\mu(\llbracket\sigma\rrbracket)$ as $\mu(\sigma)$ for strings σ. We also denote the *Lebesgue measure* by λ, where $\lambda(\sigma) = 2^{-|\sigma|}$ for $\sigma \in 2^{<\omega}$. For a computable measure μ on 2^ω, recall that a *μ-Martin-Löf test* is a sequence $(\mathcal{U}_i)_{i\in\omega}$ of uniformly effectively open subsets of 2^ω such that for each i,

$$\mu(\mathcal{U}_i) \leq 2^{-i}.$$

Moreover, $X \in 2^\omega$ *passes* the μ-Martin-Löf test $(\mathcal{U}_i)_{i\in\omega}$ if $X \notin \bigcap_{i\in\omega} \mathcal{U}_i$. Lastly, $X \in 2^\omega$ is *μ-Martin-Löf random*, denoted $X \in \mathsf{MLR}_\mu$, if X passes every μ-Martin-Löf test. When μ is the Lebesgue measure λ, we often abbreviate MLR_μ by MLR. We say that X is *μ-Martin-Löf random relative to* $A \in 2^\omega, 3^\omega$ if the sequence $(\mathcal{U}_i)_{i\in\omega}$ is uniformly effectively open relative to A.

A *labeling* for a continuous functional F is a function $\ell : 2^{<\omega} \to \{0, 1, B\}$, where we use B to denote 2 (corresponding to "blank"). A labeling ℓ induces a representation $f_\ell(\sigma)$ given by deleting all of the blank symbols from $(\ell(\sigma{\upharpoonright}i))_{i\leq|\sigma|}$. The representation induced by a labeling induces a functional F_ℓ, and every continuous functional F has some labeling ℓ such that $F = F_\ell$. We code labelings ℓ by sequences $x \in 3^\omega$, and then define a *μ-random continuous functional* to be a

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continuous functional F with a code $x \in \mathsf{MLR}_\mu$ such that $F = F_{\ell_x}$. Note that an $x \in 3^\omega$ codes not only a labeling ℓ_x and a functional $F_x = F_{\ell_x}$, but also a representation $f_x = f_{\ell_x}$.

A Bernoulli measure on 3^ω is given by parameters p_0, p_1, p_2 such that for any n, the probability that $x(n) = i$ is given by p_i.

For a representation f, the *f-output/input ratio* of σ, $\mathrm{OI}_f(\sigma)$, is defined to be

$$\mathrm{OI}_f(\sigma) = \frac{|f(\sigma)|}{|\sigma|}.$$

We write $\mathrm{OI}_f(X)$ for $\limsup_{n\to\infty} \mathrm{OI}_f(X{\upharpoonright}n)$; we refer to this as the *f-extraction rate along X*. This extraction rate depends on the particular representation f for a continuous functional F. However, we use a particular representation to define a canonical extraction rate for F, the *canonical representation* ϕ_F, defined by setting $\phi_F(\sigma)$ to be the longest common initial segment of the members of $\{F(X) : X \succ \sigma\}$. We then define the quantity $\mathrm{OI}_F(X)$ to be $\mathrm{OI}_{\phi_F}(X)$.

The *average F-output/input ratio* for strings of length n with respect to μ, denoted $Avg(F, \mu, n)$, is defined to be

$$Avg(F, \mu, n) = \sum_{\sigma \in 2^n} \mu(\sigma)\mathrm{OI}_F(\sigma).$$

We consider the behavior of this average in the limit, which leads to the following definition: the μ-extraction rate of F is defined to be

$$Rate(F, \mu) = \limsup_{n\to\infty} Avg(F, \mu, n).$$

We may now state the main results:

Theorem. *Let μ be a computable Bernoulli measure on 3^ω with parameters p_0, p_1, p_2 and let F be a μ-random continuous functional with code $x \in \mathsf{MLR}_\mu$. Let $A \in 2^\omega$ be Martin-Löf random relative to x. Then we have the following:*

(i) $Rate(F, \lambda) = p_0 + p_1$;

(ii) $\lim_{n\to\infty} \frac{|\phi_F(A{\upharpoonright}n)|}{|f_x(n)|} = 1$; *and*

(iii) $\lim_{n\to\infty} \mathrm{OI}_F(A{\upharpoonright}n) = Rate(F, \lambda)$.

References

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