REDUNDANCY OF INFORMATION: LOWERING EFFECTIVE DIMENSION

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The (effective Hausdorff) dimension of an infinite binary sequence Y is a real number in $[0,1]$ which measures the asymptotic information density of Y (Lutz [2], Mayordomo [3]). Given some Y of dimension s, how much of Y's bits do we need to change in order to obtain some X of dimension t? Here, the proportion of bits we change to go from Y to X is measured using the Besicovitch pseudo-metric d:

$$d(X,Y) = \limsup_{n\to\infty} \frac{|\{m < n : X(m) \neq Y(m)\}|}{n} \in [0,1].$$

The above question was first studied by Greenberg, Miller, Shen and Westrick [1]. In order to describe our results, we shall summarize some of their results. Henceforth we always have $\dim(Y) = s$ and $\dim(X) = t$.

(1) $d(X,Y)$ is always at least $H^{-1}(|s-t|)$, where H is the binary entropy function.
(2) If $t \geq s$, then for every Y, there is some X such that $d(X,Y) \leq H^{-1}(t) - H^{-1}(s)$.
(3) (2) is optimal: There is some Y (e.g., any Bernoulli $H^{-1}(s)$-random) such that every X satisfies $d(X,Y) = H^{-1}(t) - H^{-1}(s)$.
(4) If $s = 1$ (and so $t \leq s$), then for every Y, there is some X such that $d(X,Y) \leq H^{-1}(1-t)$.
(5) (4) is optimal by (1): For every Y and every X, we have $d(X,Y) = H^{-1}(1-t)$.

In [1], they also showed that the naive generalization of (4) to $s < 1$ is false, e.g., there are $t < s < 1$ and some Y such that for all X, we have $d(X,Y) > H^{-1}(s-t)$. We provide the correct generalization of (4) to $s < 1$:

Theorem 1. *Fix $t \leq s$. For every Y (of dimension s), there is some X (of dimension t) such that $d(X,Y) \leq \mathrm{Worst}(s,t)$, where*

$$\mathrm{Worst}(s,t) := \begin{cases} H^{-1}(1-t) & \text{if } t \leq 1 - H(2^{s-1}) \\ \frac{1-2^{s-1}}{s-(1-H(2^{s-1}))}(s-t) & \text{otherwise} \end{cases}.$$

(See Figure 1.) Furthermore, the above bound is optimal: There is some Y (of dimension s) such that for every X (of dimension t), we have $d(X,Y) = \mathrm{Worst}(s,t)$.

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Therefore, for small values of t, the upper bound given by [1] ((4) above) is optimal. But as t gets closer to s, a better strategy for constructing X becomes available. Roughly speaking, while constructing X from Y, our strategy is to alternate between copying intervals from Y (thereby keeping $d(X,Y)$ low) and making changes to intervals from Y in order to lower the dimension (thereby ensuring $\dim(X) = t$).

To prove optimality in Theorem 1, we consider a class of Y which we call *s-codewords*: Such Y are constructed in order to maximize the redundancy of their information. Not only are s-codewords at a maximum distance from sequences of dimension $t < s$ (i.e., distance $\mathrm{Worst}(s,t)$), we show that they achieve minimum distance with some sequence of dimension $t > s$:

Proposition 2. *Fix $t \geq s$. For every s-codeword Y, there is some X (of dimension t) such that $d(X,Y) = H^{-1}(t-s)$ (cf. (1).)*

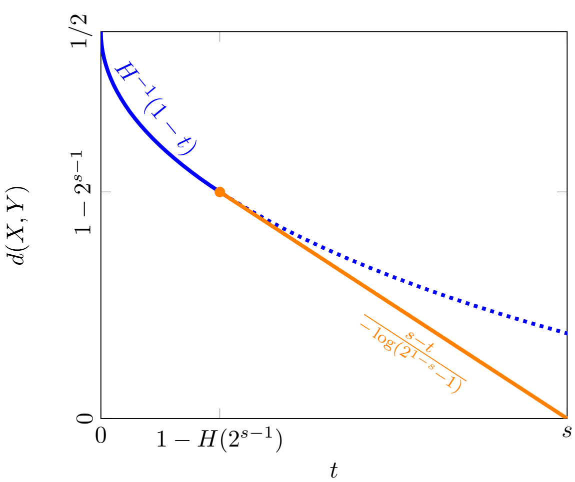

FIGURE 1. Graph of $\mathrm{Worst}(s,t)$ for $t < s = 1/2$.

REFERENCES

[1] Noam Greenberg, Joseph S. Miller, Alexander Shen, and Linda Brown Westrick. Dimension 1 sequences are close to randoms. *Theoret. Comput. Sci.*, 705:99–112, 2018.

[2] Jack H. Lutz. Gales and the constructive dimension of individual sequences. In *Automata, languages and programming (Geneva, 2000)*, volume 1853 of *Lecture Notes in Comput. Sci.*, pages 902–913. Springer, Berlin, 2000.

[3] Elvira Mayordomo. A Kolmogorov complexity characterization of constructive Hausdorff dimension. *Inform. Process. Lett.*, 84(1):1–3, 2002.

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