FIXED POINTS OF POSITIVE FORMULAE AND OF MONOTONE FUNCTIONS (EXTENDED ABSTRACT)

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We consider the language $\mathcal{L}(\tilde{R})$, which is obtained from the language $\mathcal{L}$ of first-order arithmetic and a unary relation symbol $\tilde{R}$. A formula $\psi(x_1,\dots,x_n,\tilde{R})$ in $\mathcal{L}(\tilde{R})$ is **positive** in $\tilde{R}$ or simply positive if $\tilde{R}$ does not appear in ψ; or it has one of the following forms: $0 \in \tilde{R}$, $1 \in \tilde{R}$, $x_i \in \tilde{R}$, $(x_i+1) \in \tilde{R}$, $x_i + x_j \in \tilde{R}$, $x_i \cdot x_j \in \tilde{R}$, $\varphi \vee \chi$, $\varphi \ \& \ \chi$, $\exists x_{n+1}\varphi(x_1,\dots,x_n,x_{n+1},\tilde{R})$, $\forall x_{n+1}\varphi(x_1,\dots,x_n,x_{n+1},\tilde{R})$, where φ and χ are positive.

Evidently a formula $\psi(x,\tilde{R})$ in $\mathcal{L}(\tilde{R})$ induces the operation

$$\Phi_\psi : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N}) : A \mapsto \{y \mid \psi(y,A) \text{ holds}\}$$

and if ψ is positive it is easy to see that Φ_ψ is **monotone**, i.e., if $A \subseteq B \subseteq \omega$ then $\Phi_\psi(A) \subseteq \Phi_\psi(B)$. A **fixed point** of ψ is a set $Q \subseteq \omega$ such that for all $y \in \omega$ we have

$$y \in Q \iff \psi(y,Q) \text{ holds},$$

equivalently Q is a fixed point of the associated operation Φ_ψ. As it is well-known a monotone operation Φ has a fixed point (see for example [3, 7C]).

The topic of fixed points in the Kripke-Platek set theory or in weak fragments of second-order arithmetic has received attention from Gerhard Jäger, Silvia Steila and Dieter Probst, see [2] and [1]. G. Jäger had asked if there exists a positive arithmetical formula with no hyperarithmetical fixed points. This was answered affirmatively by D. Probst in his Thesis.

Here we construct another such formula, which has the property that every of its fixed points contains uniformly-recursively all hyperarithmetical (*HYP*) subsets of $\mathbb{N}$. To explain this we employ the following natural encoding of *HYP*. We denote by $\{e\}$ the e-th largest partial recursive function on $\mathbb{N}$ to $\mathbb{N}$ and we consider a recursive injective function $\langle\cdot\rangle : \mathbb{N}^2 \to \mathbb{N}$.

We define by recursion the family $(I_\xi)_{\xi<\omega_1^{CK}}$ of subsets of $\mathbb{N}$ as follows

$$\begin{cases} I_0 = \{\langle 0,e\rangle \mid e \in \mathbb{N}\} \\ I_\xi = \{\langle 1,e\rangle \mid e \in \mathbb{N} \ \& \ (\forall k)(\exists \eta < \xi)[\{e\}(k) \in I_\eta]\}. \end{cases}$$

Further for $a \in \bigcup_{\xi<\omega_1^{CK}} I_\xi$ we define the sets

$$\mathrm{Hp}(a) = \begin{cases} \text{the singleton } \{e\}, & \text{if } a = \langle 0,e\rangle, \\ \bigcup_t \bigcap_s \mathrm{Hp}(\{e\}(\langle t,s\rangle)), & \text{if } a = \langle 1,e\rangle. \end{cases}$$

Every *HYP* subset of $\mathbb{N}$ is of the form $\mathrm{Hp}(a)$ for some $a \in I$. In fact this holds **uniformly in the codes**. Then we have the following result.

Theorem 1. *Consider the positive formula of $\mathcal{L}(\tilde{R})$ defined by*

$$\psi(y,\tilde{R}) \equiv (\exists a,x,e)\big\{y=\langle a,x\rangle \ \& \ \big(\ [a=\langle 0,e\rangle \ \& \ x=e] \\ \lor [a=\langle 1,e\rangle \ \& \ (\exists t)(\forall s)[\langle\{e\}(\langle t,s\rangle),x\rangle \in \tilde{R}] \ \big)\big\}.$$

Then for every fixed point Q of ψ, for all $a\in I$ and all $x\in\mathbb{N}$ we have

$$x \in \mathrm{Hp}(a) \iff \langle a,x\rangle \in Q.$$

In particular ψ has no hyperarithmetical fixed points.

The preceding result is straightforward to **relativize** with respect to a parameter. This in turn has consequences to the non Borel uniformization of certain Borel sets in classical descriptive set theory. Below we identify $\mathcal{P}(\mathbb{N})$ with ${}^{\mathbb{N}}2$.

Corollary 2. *There exists a function $f : {}^{\mathbb{N}}2\times{}^{\mathbb{N}}2 \to {}^{\mathbb{N}}2$ with the following properties:*

(*i*) *each section $f_\gamma : ({}^{\mathbb{N}}2,\subseteq) \to ({}^{\mathbb{N}}2,\subseteq)$ is monotone and therefore it has a fixed point;*
(*ii*) *the function f is $\utilde{\mathbf{\Sigma}}^0_4$-measurable;*
(*iii*) *there is no Borel-measurable function $u : {}^{\mathbb{N}}2 \to {}^{\mathbb{N}}2$ such that $u(\gamma)$ is a fixed point of f_γ for all $\gamma \in {}^{\mathbb{N}}2$.*

In particular the set

$$P = \{(\gamma,\alpha) \in {}^{\mathbb{N}}2 \times {}^{\mathbb{N}}2 \mid f(\gamma,\alpha)=\alpha\}$$

is $\utilde{\mathbf{\Pi}}^0_4$, has non-empty sections $P(\gamma)$ for all $\gamma \in {}^{\mathbb{N}}2$, and cannot be uniformized by any Borel set.

For compactness reasons no function that satisfies Corollary 2 can be recursive. We believe that if we replace the Cantor space with the Baire space and with a necessary modification of $\subseteq$ to a Π^0_1 relation on ${}^{\mathbb{N}}\mathbb{N}$ we can indeed obtain a recursive function f as above.

Conjecture 3. *There exists a Π^0_1 partial ordering $\preceq$ on ${}^{\mathbb{N}}\mathbb{N}$ such that every $\preceq$-chain has a least upper bound and a function $f : {}^{\mathbb{N}}\mathbb{N} \times {}^{\mathbb{N}}\mathbb{N} \to {}^{\mathbb{N}}\mathbb{N}$ with the following properties:*

(*i*) *the function f is recursive;*
(*ii*) *each section $f_\gamma : ({}^{\mathbb{N}}\mathbb{N},\preceq) \to ({}^{\mathbb{N}}\mathbb{N},\preceq)$ is monotone and therefore it has a fixed point;*
(*iii*) *there is no Borel-measurable function $u : {}^{\mathbb{N}}2 \to {}^{\mathbb{N}}2$ such that $u(\gamma)$ is a fixed point of f_γ for all $\gamma \in {}^{\mathbb{N}}2$.*

References

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