Further Properties of Nearly Computable Real Numbers

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A real number x is called *computable* if there exists a computable sequence $(q_n)_n$ of rational numbers with $|x - q_n| \leq 2^{-n}$, for all n. From a computability-theoretic point of view the set $\mathbb{R}_c$ of computable real numbers is certainly the most important subset of the field of real numbers. But there are also larger subsets that are of interest from a computability-theoretic point of view. For example, a real number x is called *left-computable* if there exists a strictly increasing computable sequence of rational numbers that converges to x. The set $\mathbb{R}_{lc}$ of left-computable real numbers plays an important role in the theory of algorithmic randomness; see [1, 6].

In [4] and [5] a computability-theoretic set of real numbers was considered that is only slightly larger than the set $\mathbb{R}_c$ of computable real numbers. In this article we discuss some further interesting properties of this number class. Let us define it. We call a sequence $(a_n)_n$ of real numbers *nearly computably Cauchy* if it has the property that for every strictly increasing computable function $r : \mathbb{N} \to \mathbb{N}$ the sequence $|a_{r(n+1)} - a_{r(n)}|$ converges computably to 0 (a sequence $(y_n)_n$ of real numbers *converges computably to* 0 if there exists a computable function $m : \mathbb{N} \to \mathbb{N}$ such that, for all $k, n \in \mathbb{N}$ with $k \geq m(n)$, $|y_k| \leq 2^{-n}$). We call a real number α *nearly computable* if there exists a computable sequence $(a_n)_n$ of rational numbers that converges to α and is nearly computably Cauchy. Let $\mathbb{R}_{nc}$ denote the set of all nearly computable real numbers.

The idea behind this is the following. The limit of any convergent sequence $(a_n)_n$ of real numbers can be written as the limit of the series $\sum_{i=0}^{\infty} b_i$, where $b_i := a_i - a_{i-1}$, for $i > 0$, and $b_0 := a_0$. Then the sequence $(b_n)_n$ converges to zero. We are interested in the computability-theoretic properties of the real numbers that one obtains as the limit of such a series when one imposes conditions on the convergence of the sequence $(b_n)_n$. Here we consider a strong condition by demanding that even for any strictly increasing computable function $r : \mathbb{N} \to \mathbb{N}$ the sequence $(\sum_{i=r(n)+1}^{r(n+1)} b_i)_n$ converges to zero computably.

We note that it follows from a result by Downey and LaForte [3] that there exists a nearly computable and left-computable number that is not computable. In [4] it was shown that the set $\mathbb{R}_{nc}$ of nearly computable real numbers is a real closed field. In fact, it is closed under computable functions with open domain. But we show that it is not closed under arbitrary computable functions. We construct a computable and strictly increasing function $f : \mathbb{R} \setminus \mathbb{Q} \to \mathbb{R}$ defined on all irrational numbers such that for every noncomputable real number α the number $f(\alpha)$ is not nearly computable.

At the conference CCA 2018, after the presentation of [4], M. Schröder had asked the following question: is every nearly computably Cauchy sequence necessarily a Cauchy sequence? On the one hand, we show that this is true for nearly computably Cauchy sequences that are additionally computable. On the other hand, we show that there exists a strictly increasing sequence of rational numbers that is nearly computably Cauchy and unbounded.

We also discuss several cases where the requirement that a real number should be nearly computable and left-computable and satisfy some additional condition forces the real number to be even computable. For example, an argument by Downey, Hirschfeldt, and LaForte [2, Pages 105, 106] shows that any strongly left-computable and nearly computable real number is even computable. Stephan and Wu [7] have shown two further results of the same type. They have shown that any nearly computable and left-computable number that is not computable is hyperimmune (and, therefore, not Martin-Löf random) and strongly Kurtz random (and, therefore, not K-trivial). We strengthen both results by showing that in both results the assumption that the real number should be left-computable can be omitted. Finally, Downey and LaForte [3] have shown another result of the same kind. They have shown that no promptly simple set can be Turing reducible to a nearly computable real number that is left-computable. We strengthen this in the same way: the assumption that the real number should be left-computable can be omitted. So, we show that no promptly simple set can be Turing reducible to a nearly computable real number.

References

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