First-order part and sequential discontinuity

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A well-known continuity notion is that convergent sequences get mapped to convergent sequences. As spaces of relevance for computable analysis are sequential, this notion adequately characterizes continuity for functions for us. Multivalued functions, on the other hand, can exhibit a very different behaviour. To one extreme, the multivalued function $\mathrm{NON} : \{0,1\}^{\mathbb{N}} \rightrightarrows \{0,1\}^{\mathbb{N}}$ with $q \in \mathrm{NON}(p)$ iff $q \not\leq_{\mathrm{T}} p$ is discontinuous, yet whenever restricted to a subset of its domain of cardinality less than the continuum even has a constant realizer. It thus seems a natural question to ask when discontinuity of a multivalued function stems from its behaviour restricted to some convergent sequence.

The answer to this turns out to be related to the notion of first-order part of a Weihrauch degree, introduced in unpublished work by Dzafahrov, Solomon and Yokoyama (see e.g. [3] for details). It is defined as:

$$^1(f) := \max_{\leq_{\mathrm{W}}}\{g :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N} \mid g \leq_{\mathrm{W}} f\}$$

It turns out that there is a weakest Weihrauch degree with non-trivial first-order part (up to continuous reductions), namely the degree of $\mathrm{ACC}_{\mathbb{N}}$ which is the restriction of $\mathrm{C}_{\mathbb{N}}$ to sets $A \subseteq \mathbb{N}$ with $|\mathbb{N} \setminus A| \leq 1$ studied in [2]. We show the following:

Theorem 1. The following are equivalent for a multi-valued function $f :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$:

1. $^1(f)$ is discontinuous.
2. There exists a convergent sequence $(a_n)_{n\in\mathbb{N}}$ with $\overline{\{a_n \mid n \in \mathbb{N}\}} \subseteq \mathrm{dom}(f)$ such that $f|_{\overline{\{a_n \mid n\in\mathbb{N}\}}}$ is discontinuous.
3. $\mathrm{ACC}_{\mathbb{N}} \leq^*_{\mathrm{W}} f$

The proof uses the generalization of Wadge-style games to multivalued functions, similar to the approaches in [4, 1].

References

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