ON THE COMPUTATIONAL PROPERTIES OF BASIC MATHEMATICAL NOTIONS

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Abstract. We investigate the computational properties of basic mathematical notions pertaining to $\mathbb{R} \to \mathbb{R}$-functions and subsets of $\mathbb{R}$, like *finiteness*, *countability*, (*absolute*) *continuity*, *bounded variation*, *suprema*, and *regularity*. We work in higher-order computability theory based on Kleene's S1-S9 schemes. We show that the aforementioned italicised properties give rise to two huge and robust classes of computationally equivalent operations, the latter based on well-known theorems from the mainstream mathematics literature. As part of this endeavour, we develop an equivalent λ-calculus formulation of S1-S9 that accommodates *partial* objects. We show that the latter are essential to our enterprise via the study of countably based and partial functionals of type 3. We also exhibit a connection to *infinite time* Turing machines.

1. Overview

Given a finite set, perhaps the most basic questions are *how many* elements it has, and *which ones*? We study this question in Kleene's *higher-order computability theory*, based on his computation schemes S1-S9 (see [2,3]). In particular, a central object of study is the higher-order functional Ω which on input a finite set of real numbers, list the elements as a finite sequence.

Perhaps surprisingly, the 'finiteness' functional Ω give rise to a *huge and robust* class of computationally equivalent operations, called the Ω-cluster, as sketched in Section 3. For instance, many basic operations on functions of *bounded variation* are part of the Ω-cluster, including those stemming from the well-known *Jordan decomposition theorem*. In addition, we identify a second cluster of computationally equivalent objects, called the Ω_1-cluster, based on Ω_1, the restriction of Ω to singletons. We also show that both clusters include basic operations on *regulated* and *Sobolev space* functions, respectively a well-known super- and sub-class of the class of BV-functions.

Our objects of study are fundamentally *partial* in nature, and we formulate an elegant and equivalent λ-calculus formulation of S1-S9 to accommodate partial objects (Section 2). The advantages of this approach are three-fold: proofs are more transparent in our λ-calculus approach, all (previously hand-waved) technical details can be settled easily, and we can show that Ω_1 and Ω are not computationally equivalent to any *total* functional.

Finally, initial results have been published as [4] while a full report of the above may be found in [5].

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Key words and phrases. Kleene S1-S9, higher-order computability theory, bounded variation, partiality, regulated functions.

2. A λ-calculus formulation of Kleene's computation schemes

Turing's 'machine' model ([7]) captures *computing with real numbers* and -by its very nature- involves partially defined objects. By contrast, Kleene's S1-S9 ([2]) is an extension of the Turing approach meant to capture *computing with objects of any finite type*, where everything is always assumed to be *total*. In this talk, we introduce an equivalent λ-calculus formulation of S1-S9 that accommodates partial objects. Our motivation for this new construct is as follows.

- We have previously hand-waved the extension of S1-S9 to partial objects, leading to confusion among our readers regarding certain technical details.
- Proofs are a lot more transparent based on fixed points (from the λ-calculus) instead of the recursion theorem (hardcoded by Kleene's S9).
- The functional Ω_1 and Ω are natural objects of study that are *fundamentally partial* in nature. Indeed, we show that these are not computationally equivalent to any *total* functional.

Finally, as an application of our new framework, we show that Ω_1 and the Suslin functional S^2 compute the halting problem for *infinite time* Turing machines.

3. Two robust clusters

The following theorems give rise to functionals in the Ω and Ω_1-clusters.

- The Jordan decomposition theorem (Jordan, 1881 [1]).
- A function of bounded variation on the unit interval has a supremum.
- For a function of bounded variation on the unit interval, there is a sequence listing all points of discontinuity.
- A countable set (=injection or bijection to $\mathbb{N}$) can be enumerated.
- A regulated function on the unit interval has a supremum.
- For a regulated function on the unit interval, there is a sequence listing all points of discontinuity.
- For regulated $f : [0,1] \to \mathbb{R}$, there are g, h such that $f = g \circ h$ with g continuous and h strictly increasing on their domains (Sierpiński, [6]).
- For a regulated function, the Banach indicatrix exists.
- A regulated function is of class Baire 1.
- A regulated upper semi-continuous function on $[0,1]$ attains a maximum

This list is by no means exhaustive, as is clear from [5, §4].

References

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