Weihrauch Reducibility on Assemblies

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April 29, 2022

Weihrauch reducibility provides a method to compare the computational content of mathematical problems [1]. The current framework formalises problems as multivalued functions between represented spaces. We discuss how to extend Weihrauch reducibility to multivalued functions between assemblies (aka. multirepresented spaces). By an assembly we mean a set endowed with a multirepresentation. Multirepresentations generalise ordinary representations by allowing any $p \in \mathbb{N}^{\mathbb{N}}$ to be a name of more than one element [2, 3]. A prominent example of a space in Computable Analysis that requires a multirepresentation is the space of partial continuous functions (see [3, Example 9.2.16]). Another example is provided by Definition 4(1).

Multifunctions and multirepresentations

A *multi-valued function* (or *multifunction*) f a triple $(X, Y, \mathrm{Graph}(f))$, where $\mathrm{Graph}(f)$ is a relation between sets X and Y, written as $f :\subseteq X \rightrightarrows Y$. The elements of $f[x] := \{y \in Y \mid (x, y) \in \mathrm{Graph}(f)\}$ are considered to be the legitimate results for an input x under f. For $h :\subseteq X \rightrightarrows Y$ one writes $f \sqsubseteq h$, if $\mathrm{dom}(f) \supseteq \mathrm{dom}(h)$ and $\forall x \in \mathrm{dom}(h). f[x] \subseteq h[x]$. A *realizer* of a multifunction f between represented spaces $\mathbf{X}$ and $\mathbf{Y}$ is a partial function F on the Baire space $\mathbb{N}^{\mathbb{N}}$ satisfying

$$\delta_{\mathbf{Y}}(F(p)) \in f[\delta_{\mathbf{X}}(p)] \quad \text{for every } p \in \mathrm{dom}(f\delta_{\mathbf{X}}), \tag{1}$$

see [1]. Here $\delta_{\mathbf{X}}$ and $\delta_{\mathbf{Y}}$ denote the respective representations of $\mathbf{X}$ and $\mathbf{Y}$.

A *multirepresentation* δ of a set X is a partial multifunction from $\mathbb{N}^{\mathbb{N}}$ to X which is surjective in the sense that for every $x \in X$ there exists some "name" $p \in \mathbb{N}^{\mathbb{N}}$ such that $x \in \delta[p]$. In this case (X, δ) is called a *multirepresented space* or *assembly*. A partial function F on $\mathbb{N}^{\mathbb{N}}$ satisfying

$$\delta_{\mathbf{Y}}[F(p)] \ni f(x) \quad \text{for all } p \in \mathrm{dom}(\delta_{\mathbf{X}}) \text{ and } x \in \delta_{\mathbf{X}}[p] \cap \mathrm{dom}(f) \tag{2}$$

is called a *realizer* for a partial function $f :\subseteq \mathbf{X} \to \mathbf{Y}$ between assemblies $\mathbf{X}, \mathbf{Y}$ (see [2, 3]).

Lifting Weihrauch reducibility to multirepresented spaces

The only reasonable way to define realizers for *multi*functions between *multi*represented spaces in such a manner that (1) and (2) are generalised seems to be the following:

Definition 1 A partial function F on $\mathbb{N}^{\mathbb{N}}$ is a *realizer* for a multifunction $f :\subseteq \mathbf{X} \rightrightarrows \mathbf{Y}$ on multirepresented spaces $\mathbf{X}, \mathbf{Y}$, if $\delta_{\mathbf{Y}}[F(p)] \cap f[x] \neq \emptyset$ for all $x \in \delta_{\mathbf{X}}[p] \cap \mathrm{dom}(f)$. In this case, we write $F \vdash f$. If f has a computable realizer, then f is called *computable*.

This means that for any name p of any $x \in \mathrm{dom}(f)$, $F(p)$ must be a name of some legitimate result $y \in f[x]$ for x. But one does not get to know which element of $\delta_{\mathbf{Y}}[F(p)]$ is a result for x. The reader should be warned that a function between assemblies need not have a realizer. In spirit of [1, Definition 11.3.1 and Proposition 11.3.2] we now define four reducibility relations as possible candidates for a generalisation of Weihrauch reducibility.

Definition 2 Let $f :\subseteq \mathbf{X} \rightrightarrows \mathbf{Y}$, $g :\subseteq \mathbf{A} \rightrightarrows \mathbf{B}$ be problems between assemblies.

(1) Define $f \leq_{\mathrm{rW}} g$ iff there are computable functions $H :\subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ and $K :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that $H(\mathrm{id}, GK) \vdash f$.

(2) Define $f \leq_{\mathrm{srW}} g$ iff there are computable functions $H, K :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that $G \vdash g \Rightarrow HGK \vdash f$.

(3) Define $f \leq_{\mathrm{aW}} g$ iff there are an assembly $\mathbf{V}$ and computable multifunctions $k :\subseteq \mathbf{X} \rightrightarrows \mathbf{V} \times \mathbf{A}$, $h :\subseteq \mathbf{V} \times \mathbf{B} \rightrightarrows \mathbf{Y}$ such that $h \circ (id_{\mathbf{V}} \times g) \circ k \sqsubseteq f$.

(4) Define $f \leq_{\mathrm{saW}} g$ iff there are computable multifunctions $k :\subseteq \mathbf{X} \rightrightarrows \mathbf{A}$ and $h :\subseteq \mathbf{B} \rightrightarrows \mathbf{Y}$ such that $h \circ g \circ k \sqsubseteq f$.

These relations are reflexive and transitive. Moreover, $f \leq_{\mathrm{srW}} g \Rightarrow f \leq_{\mathrm{rW}} g$ and $f \leq_{\mathrm{saW}} g \Rightarrow f \leq_{\mathrm{aW}} g$. Our first result states that the abstract versions $\leq_{\mathrm{aW}}$ and $\leq_{\mathrm{saW}}$ of ordinary and strong Weihrauch reduciblity are strictly finer than their realizer-based counterparts $\leq_{\mathrm{rW}}$ and $\leq_{\mathrm{srW}}$.

Proposition 3

(1) *$f \leq_{\mathrm{aW}} g$ implies $f \leq_{\mathrm{rW}} g$, and $f \leq_{\mathrm{saW}} g$ implies $f \leq_{\mathrm{srW}} g$.*

(2) *There are single-valued problems c, d such that $c \leq_{\mathrm{srW}} d$ and $c \not\leq_{\mathrm{aW}} d$.*

Multirepresentations allow us to represent the powerset of a (multi-)represented space.

Definition 4 *Let $\mathbf{X}, \mathbf{Y}$ be assemblies.*

(1) *Denote by $\mathbf{P}(\mathbf{Y})$ the space of non-empty subsets of $\mathbf{Y}$ equipped with the multirepresentation defined by $\delta_{\mathbf{P}(\mathbf{Y})}[q] \ni M :\Leftrightarrow \delta_{\mathbf{Y}}[q] \cap M \neq \emptyset$.*

(2) *For a problem $f :\subseteq \mathbf{X} \rightrightarrows \mathbf{Y}$ define $f^{\mathbf{P}} :\subseteq \mathbf{X} \to \mathbf{P}(\mathbf{Y})$ by $f^{\mathbf{P}}(x) := f[x]$ for all $x \in \mathrm{dom}(f)$.*

The function $f^{\mathbf{P}}$ relates to f as follows:

Proposition 5 *For any $f :\subseteq \mathbf{X} \rightrightarrows \mathbf{Y}$ we have $f \equiv_{\mathrm{srW}} f^{\mathbf{P}}$ and $f \leq_{\mathrm{saW}} f^{\mathbf{P}} \leq_{\mathrm{aW}} f$.*

The concept of considering Weihrauch reducibility on assemblies has as one benefit that any Weihrauch degree (induced by $\equiv_{\mathrm{rW}}$ or by $\equiv_{\mathrm{aW}}$) contains a single-valued function by Proposition 5. In general, $\mathbf{P}(\mathbf{Y})$ is not admissible, even if $\mathbf{Y}$ is. It would be interesting to know whether any non-empty Weihrauch degree contains a single-valued function between spaces with an admissible multirepresentation in the sense of [2]. A basic open question is which of the relations $\leq_{\mathrm{rW}}$ and $\leq_{\mathrm{aW}}$ provides a better extension of the traditional Weihrauch reducibility relation $\leq_{\mathrm{W}}$.

References

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