Nondeterministic limits and certified exact real computation*

Michal Konečný[1], Sewon Park[2], and Holger Thies[2]

[1]Aston University, UK [2]Kyoto University, Japan

In [KPT21a], we propose an axiomatization of real numbers in a dependent type theory which formalizes real numbers in a conceptually similar way as some mature implementations of exact real computation and implement our theory in the Coq proof assistant. We use Coq's code extraction tools to extract Haskell programs from proofs. The extracted programs use the AERN framework [Kon21] to realize basic operations on real numbers. Our Coq implementation and the extracted Haskell/AERN programs can be found in `https://github.com/holgerthies/coq-aern` . A first version of our work has already been presented at CCA 2021 [KPT21b].

There, we model nondeterminism by a monad such that for any type X : Type, we have a new type $\mathsf{M}X : \mathsf{Type}$ that stands for the type of nondeterministic computations in X. The monad can be used to define various forms of nondeterminism that are common in computable analysis, for example, $f : A \to \mathsf{M}B$ denotes a computable multivalued function $A \rightrightarrows B$, $\mathsf{M}(A + B)$ denotes a nondeterministic decision procedure deciding either A or B, and $\mathsf{M}\Sigma(x : A).\ B$ denotes the nondeterminstic existence of a $x : A$ such that $B(x)$.

Combining this kind of nondeterminism with the ability to compute limits of rapidly converging sequences becomes subtle; c.f., [FB18, Kon18]. In general, we can distinguish three distinct cases where we need to compute limits:

(1) a sequence of real numbers converge to a deterministic point,

(2) a sequence of nondeterministic real numbers converge to a point, and

(3) a sequence of nondeterministic real numbers converge to a nondeterministic point.

The first case is of course just the ordinary metric completeness which is realized by primitive limit operations in exact real number computation software, mapping fast Cauchy sequences to their limits. We derive appropriate versions for the second and third case from a more general principle, the *nondeterministic dependent choice principle*.

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Given a sequence of types $P : \mathsf{N} \to \mathsf{Type}$ and a sequence of *classical* binary relations $R_n : (P\, n) \to (P\,(n+1)) \to \mathsf{Prop}$, the type

$$\Pi(n : \mathsf{N}).\ \Pi(x : P\, n).\ \mathsf{M}\Sigma(y : P\,(n+1)).\ R_n\, x\, y$$

represents the space of nondeterministic procedures f such that when $x : P\, n$, $f\, n\, x$ nondeterministically computes $y : P\,(n+1)$ and a classical reason why $R_n\, x\, y$ holds. The nondeterministic dependent choice says that when we have such f, we get a term of type

$$\mathsf{M}\Sigma(g : \Pi(n : \mathsf{N}).\ P\, n).\ \Pi(m : \mathsf{N}).\ R\, m\,(g\, m)\,(g\,(m+1))$$

satisfying a certain coherence condition. In words: when we have a procedure that nondeterministically chooses the second entry y of $R_n\, x\, y$, we nondeterministically get a sequence $x_0, x_1, \cdots$ that entry-wise satisfies the binary relation $R_n\, x_n\, x_{n+1}$.

The principle can for example be used to derive the limit operations suggested in [FB18], [Mül00, Section 10.3], and some other forms of nondeterministic limits that are useful in applications. We further give several natural examples where nondeterministic limits can be used in exact real computation and extract AERN programs from these examples. A full version describing the results is available [KPT22].

References

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