Borel Combinatorics Goes Wrong in HYP

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Although Reverse Mathematics is typically done in second order theories, which contain only natural numbers and sets of natural numbers, it is possible to talk about Borel sets of real numbers by a suitable encoding: a Borel set can be represented by a well-founded tree encoding the inductive construction of the Borel set as a countable sequence of unions and intersections from open sets.

Determining whether a given real number belongs to the set described by such a code requires a witness describing inductively how the real number belongs in each of the sets constructed along the way to the given set. One challenge to investigating the reverse mathematics of theorems involving the Borel sets is that proving that such witnesses always exist already requires the fairly strong theory ATR_0, which also proves many substantive theorems about the Borel sets.

In this setting, [1] introduces the notion of a *completely determined Borel set*: a code for what looks like a Borel set which happens to have the property that all the needed witnesses exist. We can then study the behavior of these sets in weaker theories of reverse math.

We study [2] the behavior of these sets in the smallest reasonable model, HYP, the collection with exactly the hyperarithmetic sets. This model contains pseudo-well-orderings—linear orders which are not well-founded, but which appear to be well-founded within the model because there are no hyperarithmetic infinite descending sequences. The "decorating trees" method introduced in [1] lets us convert these pseudo-well-orderings into ill-founded trees which look, within HYP, like completely determined Borel sets.

We show that the completely determined Borel sets in this model are precisely the $\Delta_1(L_{\omega_1^{ck}})$ sets and use this to show that the completely determined Borel sets of HYP behave more like computable sets than like true Borel sets: there is a Borel well-ordering of the reals, every connected Borel graph without odd cycles has a Borel 2-coloring, and the Borel Dual Ramsey Theorem fails. Many of these constructions have a flavor reminiscent of computable constructions:

Joint work with Rose Weisshaar and Linda Westrick.

References

[1] E. P. Astor et al. "The determined property of Baire in reverse math". In: *J. Symb. Log.* 85.1 (2020), pp. 166–198. ISSN: 0022-4812 (cit. on p. 1).

[2] H. Towsner, R. Weisshaar, and L. Westrick. *Borel combinatorics fail in HYP*. 2021. eprint: `arXiv:2106.13330` (cit. on p. 1).