Department of Mathematics, University of Udine

Embeddability of graphs and Weihrauch degrees

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An undirected graph $G_0 = (V_0, E_0)$ is a *subgraph* of $G = (V, E)$ if and only if $V_0 \subseteq V$ and $E_0 \subseteq E$. We say it is an *induced subgraph* if furthermore $E_0 = E \cap (V_0 \times V_0)$.

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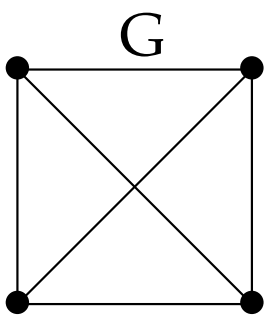

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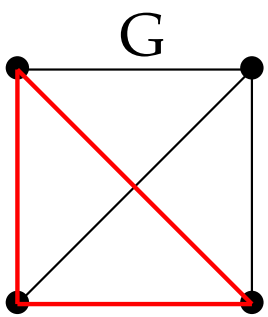

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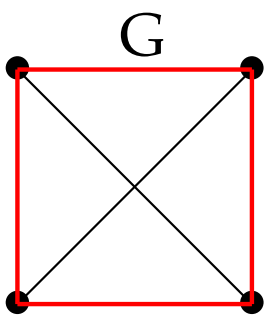

G_0 is a subgraph of G, but not an induced one.

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In this talk, using the framework of *Weihrauch reducibility* we study the uniform computational content of the (induced) subgraph problem. The firsts to consider these problems in this context were BeMent, Hirst and Wallace in [BHW21].

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A represented space $\mathbf{X}$ is a pair $(X, \delta_{\mathbf{X}})$ where $\delta_{\mathbf{X}} :\subseteq \mathbb{N}^{\mathbb{N}} \to X$.

If $\delta_{\mathbf{X}}(p) = x$ for $p \in \mathbb{N}^{\mathbb{N}}$, we say that p is a *name* for $x \in X$, ($\subseteq$ indicates that $\delta_{\mathbf{X}}$ is partial).

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Notation: **Gr** is the represented space of countable undirected graphs, i.e. p is a name for a graph G iff

- $p(\langle i, i\rangle) = 1$ iff $v_i \in V(G)$ (i.e, v_i is a vertex of G) and
- for $i \neq j$, $p(\langle i, j\rangle) = 1$ iff $v_i, v_j \in V(G)$ and $(v_i, v_j) \in E(G)$ (i.e. (v_i, v_j) is an edge of G).

Weihrauch reducibility

We denote a *multi-valued function* between represented spaces $\mathbf{X}$ and $\mathbf{Y}$ by $f :\subseteq \mathbf{X} \rightrightarrows \mathbf{Y}$. ($\rightrightarrows$ denotes that $x \in \mathbf{X}$ may be mapped to different elements of $\mathbf{Y}$).

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Let f, g be (partial multivalued) functions.
f is *Weihrauch reducible* to g ($f \leq_{\mathrm{W}} g$) if there are computable $\Phi, \Psi :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ s.t.:

- Given a name p for $x \in \mathrm{dom}(f)$, $\Phi(p)$ is a name for $z \in \mathrm{dom}(g)$;
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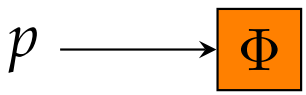

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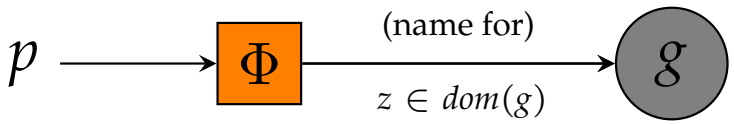

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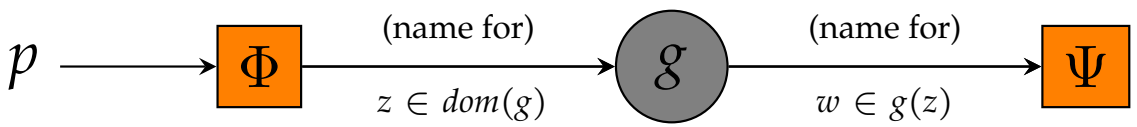

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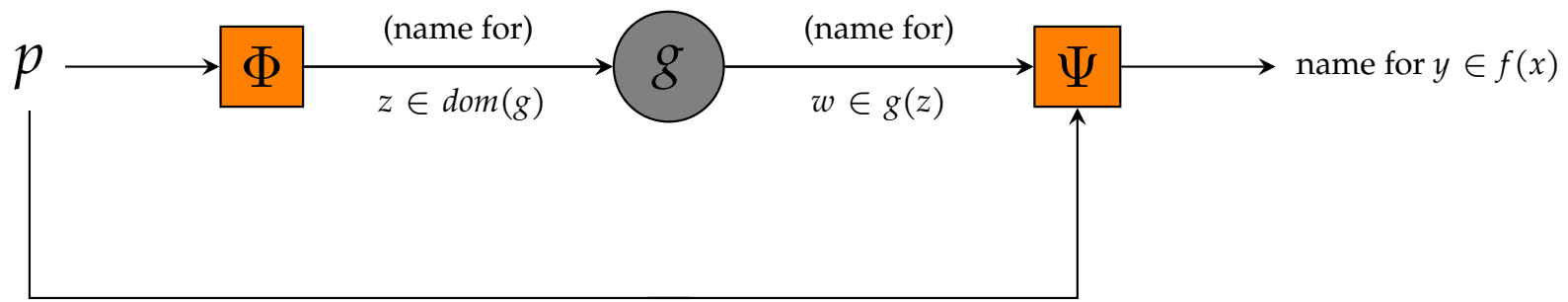

LPO and WF

If $f \leq_W g$ and viceversa we write $f \equiv_W g$. The $\equiv_W$-equivalence classes are called *Weihrauch degrees*.

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The following principles are important for the rest of the talk:

- LPO : given in input $p \in 2^{\mathbb{N}}$, $\mathsf{LPO}(p) = 1$ iff $p = 0^{\mathbb{N}}$ (answers a $\mathbf{\Sigma}^0_1$ or $\mathbf{\Pi}^0_1$ question).
- WF: Given in input a tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$, $\mathsf{WF}(T) = 1$ if T is well-founded (i.e. has no infinite branches), 0 otherwise (answers a $\mathbf{\Pi}^1_1$ or $\mathbf{\Sigma}^1_1$ question).

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Notation: Given two graphs G and H,

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In [BHW21], the authors, using a different notation, defined the functions $\mathsf{IS}_G : \mathbf{Gr} \to 2$ and $\mathsf{S}_G : \mathbf{Gr} \to 2$ for some fixed graph G as

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Theorem ([BHW21])

$\mathsf{WF} \equiv_W \mathsf{IS}_{\mathsf{R}} \equiv_W \mathsf{S}_{\mathsf{R}}$ *and if* G *is finite,* $\mathsf{LPO} \equiv_W \mathsf{IS}_G \equiv_W \mathsf{S}_G$.

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Lemma ((2) C., Pauly)

Let K_ω be the complete graph on ω vertices and let G be an infinite computable graph s.t. $K_\omega \not\subseteq_{\mathbf{s}} G$. Then $\mathsf{IS}_G \equiv_W \mathsf{WF}$.

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To prove $\mathsf{WF} \leq_{\mathsf{W}} \mathsf{IS}_G$ when $K_\omega \not\subseteq_{\mathbf{s}} G$ we proceed in the same fashion with the only difference that

$$E(H) = \{(\sigma, \tau) : (v_{|\sigma|}, v_{|\tau|}) \in E(G) \vee (\sigma|\tau)\}.$$

A negative answer

We have just shown that:

$$(1)\mathsf{R} \subseteq_{\mathbf{s}} G \implies \mathsf{IS}_G \equiv_{\mathsf{W}} \mathsf{WF} \text{ and } (2)K_\omega \not\subseteq_{\mathbf{s}} G \implies \mathsf{IS}_G \equiv_{\mathsf{W}} \mathsf{WF}.$$

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What about S_G? We first define the *jump* of a problem. Given a problem f we define its jump f' as f with the only difference that f' receives in input a name that converges to an input of f. Similarly, we can define the n-fold jump $f^{(n)}$ of a problem f (notice that $f^{(0)} = f$).

Infinite and computable G implies LPO'

In [BHW21] the authors already showed that $\mathsf{S}_{\mathsf{R}} \equiv_{\mathsf{W}} \mathsf{WF}$, and, if G is finite, $\mathsf{S}_G \equiv_{\mathsf{W}} \mathsf{LPO}$.

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It is convenient to consider $\mathsf{LPO}^{(n)}$ as the principle answering a $\mathbf{\Sigma}^0_{n+1}$ (or equivalently $\mathbf{\Pi}^0_{n+1}$) question.
Using the same construction used to prove that $\mathsf{S}_G \equiv_\mathsf{W} \mathsf{WF}$ with $K_\omega \not\subseteq_\mathbf{s} G$ we obtain the following.

Proposition (C., Pauly)

Let G be an infinite computable graph. Then $\mathsf{LPO}' \leq_\mathsf{W} \mathsf{S}_G$.

Is there any computable graph G s.t. $\mathsf{S}_G \equiv_\mathsf{W} f$ for $f \not\equiv_\mathsf{W} \mathsf{WF}, \mathsf{LPO}, \mathsf{LPO}'$?

S_G and $\mathsf{LPO}^{(n)}$

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We define G_n as follows:

- if $n = 2k$, $V(G_{2k}) = \{\sigma \in \mathbb{N}^{<\mathbb{N}} : |\sigma| \leq k\}$;
- if $n = 2k+1$, $V(G_{2k+1}) = \{\sigma \in \mathbb{N}^{<\mathbb{N}} : 0 < |\sigma| \leq k+1\}$.

In both cases,
$(\sigma, \tau) \in E(G_n) \iff (\sigma = \tau[|\tau| - 1] \vee \tau = \sigma[|\sigma| - 1])$.

G_n, informally

- G_0 is just a single vertex
- G_1 is the disconnected union of infinitely many copies of G_0, i.e. infinitely many disconnected vertices,
- G_2 is the graph obtained connecting to a new vertex all the components of G_1, i.e. a tree of height one with infinitely many children,
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Key observation: $G_n \subseteq_{\mathbf{s}} H$ iff $\varphi_n(H)$ where if n is even φ_n is a $\mathbf{\Sigma}^0_{n+1}$-complete formula, while if n is odd φ_n is a $\mathbf{\Pi}^0_{n+1}$-complete one.

G_n, informally

- G_0 is just a single vertex
- G_1 is the disconnected union of infinitely many copies of G_0, i.e. infinitely many disconnected vertices,
- G_2 is the graph obtained connecting to a new vertex all the components of G_1, i.e. a tree of height one with infinitely many children,
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Key observation: $G_n \subseteq_{\mathbf{s}} H$ iff $\varphi_n(H)$ where if n is even φ_n is a $\mathbf{\Sigma}^0_{n+1}$-complete formula, while if n is odd φ_n is a $\mathbf{\Pi}^0_{n+1}$-complete one. For example $G_3 \subseteq_{\mathbf{s}} H$ iff
$(\exists^\infty v \in V(H))(\exists^\infty w \in V(H))((v,w) \in E(H))$.

When G is in the graph

Finding the induced subgraph

We first introduce the following well known multi-valued function.

- $\mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$: given in input $T \subseteq \mathbb{N}^{<\mathbb{N}}$ s.t. T is ill-founded output some $p \in [T]$.

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Recall that WF answers the question whether a tree T is well-founded or not. In the previous slides we showed that $\mathsf{WF} \equiv_{\mathsf{W}} \mathsf{IS}_G$ where G is an infinite computable graph: in some sense, IS_G is the counterpart of WF in the induced subgraph terms.

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For some fixed computable graph G, let $\mathsf{FindIS}_G :\subseteq \mathbf{Gr} \rightrightarrows \mathbf{Gr}$ be s.t.

$$\mathsf{FindIS}_G(H) := \{G' : G' \cong G\}$$

where $\mathrm{dom}(\mathsf{FindIS}_G) = \{H : G \subseteq_{\mathbf{is}} H\}$.

$\mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$ and FindIS_G

Proposition (C., Pauly)

If G is computable, then $\mathsf{FindIS}_G \leq_W \mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$. In case G is also finite, FindIS_G is computable.

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Notation: $\deg^G(v) = n$ denotes "v has n-many neighbors in G".

Theorem (C., Pauly)

Let G be an infinite computable graph s.t.

- $|\{v \in V(G) : \deg^G(v) < \aleph_0\}| < \aleph_0$ *or*
- $|\{v \in V(G) : \deg^G(v) = \aleph_0\}| < \aleph_0$.

Then $\mathsf{C}_{\mathbb{N}^{\mathbb{N}}} \equiv_W \mathsf{FindIS}_G$.

Finding the induced subgraph

Notice that in the reduction $\mathsf{C}_{\mathbb{N}^{\mathbb{N}}} \leq_{\mathsf{W}} \mathsf{FindIS}_G$, the input for FindIS_G is computed as in $\mathsf{WF} \leq_{\mathsf{W}} \mathsf{IS}_G$ for $K_\omega \not\subseteq_{\mathbf{s}} G$.

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The only G's left out from the theorem above are those s.t.
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$\mathsf{FindS}_{\mathsf{R}}$

- $\lim_2$: given in input a converging sequence in $\{0,1\}$, outputs its limit.

Proposition (C., Pauly)

$\mathsf{C}_{\mathbb{N}^{\mathbb{N}}} \equiv_W \lim_2 * \mathsf{FindS}_{\mathsf{R}}$.

Here $*$ denotes the *compositional product*. For two multivalued functions $f, g, f * g$ denotes that, given a (name for an) input for g, we apply g, we compute a name for an input of f, and finally we apply f.

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The proposition above shows that $\mathsf{FindS}_{\mathsf{R}}$ needs very little to reach $\mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$, on the other hand, the question whether $\mathsf{C}_{\mathbb{N}^{\mathbb{N}}} \equiv_W \mathsf{FindS}_{\mathsf{R}}$ is still open.

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Theorem (C., Pauly)

Let G be as above:

- $(\forall^\infty i)(\exists^\infty j)(F_i \subseteq_{\mathbf{s}} F_j)$ *implies that* FindS_G *is computable.*
- *Let C_n be the cyclic graph of n vertices. Then for $G' = \bigoplus_{n \geq 3} C_n$ (notice that G' does not satisfy the condition above) we have that* $\mathsf{FindS}_{G'} \equiv_W \mathsf{C}_{\mathbb{N}^\mathbb{N}}$.

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We conclude this section mentioning that with the same graphs used to prove that for every n, $\mathsf{LPO}^{(n)} \equiv_W \mathsf{S}_{G_n}$, we obtained a similar result for FindS_{G_n}.

The opposite problem

Some results

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$$\mathsf{IS}^G = 1 \iff H \subseteq_{\mathbf{is}} G.$$

Notation: K_ω is the complete graph with infinitely many vertices, K_i is the complete graph of i vertices and D the totally disconnected graph. In [BHW21] the authors proved that $\mathsf{IS}^{K_\omega} \equiv_{\mathsf{W}} \mathsf{IS}^{D} \equiv_{\mathsf{W}} \mathsf{LPO}$ and left open the following question: is there a graph G s.t. $\mathsf{LPO} <_{\mathsf{W}} \mathsf{IS}^G$?

Theorem (C., Pauly)

Let $G = \bigoplus_{n \geq 3} K_n$. *Then* $\mathsf{IS}^G \equiv_{\mathsf{W}} \mathsf{LPO}''$.

Still open: is there a computable graph G s.t.$\mathsf{IS}^G \equiv_{\mathsf{W}} \mathsf{LPO}'$ or s.t. $\mathsf{LPO}'' <_{\mathsf{W}} \mathsf{IS}^G$?

Thanks for Your attention!

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