Oracle computability of conditional expectations onto subfactors

Isaac Goldbring

University of California, Irvine

19th International Conference on Computability and Complexity in Analysis
May 2022

A theorem of Macintyre

Theorem (Macintyre)

For any $n \geq 1$, the relation "b belongs to the subgroup generated by $a_1, \ldots, a_n$" is uniformly defined in the class of existentially closed groups by the formula

$$\varphi(x_1, \ldots, x_n, y) \equiv \forall z \left(\bigwedge_{i=1}^{n} [z, x_i] = e \rightarrow [z, y] = e \right).$$

Definition

A group G is **existentially closed** (e.c.) if: for any quantifier-free formula $\psi(\vec{x}, \vec{y})$, any $\vec{a} \in G$, and any group H containing G,

$$H \models \exists \vec{y} \psi(\vec{a}, \vec{y}) \Rightarrow G \models \exists \vec{y} \psi(\vec{a}, \vec{y}).$$

A theorem of Macintyre

Theorem (Macintyre)

For any $n \geq 1$, the relation "b belongs to the subgroup generated by $a_1, \ldots, a_n$" is uniformly defined in the class of existentially closed groups by the formula

$$\varphi(x_1, \ldots, x_n, y) \equiv \forall z \left(\bigwedge_{i=1}^{n} [z, x_i] = e \rightarrow [z, y] = e \right).$$

Definition

A group G is **existentially closed** (e.c.) if: for any quantifier-free formula $\psi(\vec{x}, \vec{y})$, any $\vec{a} \in G$, and any group H containing G,

$$H \models \exists \vec{y} \psi(\vec{a}, \vec{y}) \Rightarrow G \models \exists \vec{y} \psi(\vec{a}, \vec{y}).$$

The proof of Macintyre's theorem

- $\varphi(\vec{x}, y) \equiv \forall z \left(\bigwedge_{i=1}^{n} [z, x_i] = e \rightarrow [z, y] = e \right)$.
- It is clear that $G \models \varphi(\vec{a}, b)$ for any group G whenever $b \in \langle a_1, \ldots, a_n \rangle$.
- Now suppose that G is e.c. and $a_1, \ldots, a_n, b \in G$ are such that $b \notin \langle a_1, \ldots, a_n \rangle$. We want $G \not\models \varphi(\vec{a}, b)$.
- Consider the subgroup $G_0 := \langle a_1, \ldots, a_n, b \rangle$ of G and set $H := G *_{\langle a_1, \ldots, a_n \rangle} G_0$.
- Let b' denote the copy of b in the "right factor".
- The map $a_i \mapsto a_i$, $b \mapsto b'$ is an isomorphism between subgroups of H and so in the corresponding **HNN extension** $H_1 \supseteq H$, there is $c \in H_1$ such that $ca_ic^{-1} = a_i$ and $cbc^{-1} = b' \neq b$.
- Consequently, $H_1 \models \neg\varphi(\vec{a}, b)$ whence $G \models \neg\varphi(\vec{a}, b)$ since G is e.c.

The proof of Macintyre's theorem

- $\varphi(\vec{x}, y) \equiv \forall z \left(\bigwedge_{i=1}^{n} [z, x_i] = e \rightarrow [z, y] = e\right).$
- It is clear that $G \models \varphi(\vec{a}, b)$ for any group G whenever $b \in \langle a_1, \ldots, a_n \rangle$.
- Now suppose that G is e.c. and $a_1, \ldots, a_n, b \in G$ are such that $b \notin \langle a_1, \ldots, a_n \rangle$. We want $G \not\models \varphi(\vec{a}, b)$.
- Consider the subgroup $G_0 := \langle a_1, \ldots, a_n, b \rangle$ of G and set $H := G *_{\langle a_1, \ldots, a_n \rangle} G_0$.
- Let b' denote the copy of b in the "right factor".
- The map $a_i \mapsto a_i$, $b \mapsto b'$ is an isomorphism between subgroups of H and so in the corresponding **HNN extension** $H_1 \supseteq H$, there is $c \in H_1$ such that $ca_ic^{-1} = a_i$ and $cbc^{-1} = b' \neq b$.
- Consequently, $H_1 \models \neg\varphi(\vec{a}, b)$ whence $G \models \neg\varphi(\vec{a}, b)$ since G is e.c.

The proof of Macintyre's theorem

- $\varphi(\vec{x}, y) \equiv \forall z \left(\bigwedge_{i=1}^{n} [z, x_i] = e \to [z, y] = e\right)$.
- It is clear that $G \models \varphi(\vec{a}, b)$ for any group G whenever $b \in \langle a_1, \ldots, a_n \rangle$.
- Now suppose that G is e.c. and $a_1, \ldots, a_n, b \in G$ are such that $b \notin \langle a_1, \ldots, a_n \rangle$. We want $G \not\models \varphi(\vec{a}, b)$.
- Consider the subgroup $G_0 := \langle a_1, \ldots, a_n, b \rangle$ of G and set $H := G *_{\langle a_1, \ldots, a_n \rangle} G_0$.
- Let b' denote the copy of b in the "right factor".
- The map $a_i \mapsto a_i$, $b \mapsto b'$ is an isomorphism between subgroups of H and so in the corresponding **HNN extension** $H_1 \supseteq H$, there is $c \in H_1$ such that $ca_ic^{-1} = a_i$ and $cbc^{-1} = b' \neq b$.
- Consequently, $H_1 \models \neg\varphi(\vec{a}, b)$ whence $G \models \neg\varphi(\vec{a}, b)$ since G is e.c.

The proof of Macintyre's theorem

- $\varphi(\vec{x}, y) \equiv \forall z \left(\bigwedge_{i=1}^{n} [z, x_i] = e \rightarrow [z, y] = e \right)$.
- It is clear that $G \models \varphi(\vec{a}, b)$ for any group G whenever $b \in \langle a_1, \ldots, a_n \rangle$.
- Now suppose that G is e.c. and $a_1, \ldots, a_n, b \in G$ are such that $b \notin \langle a_1, \ldots, a_n \rangle$. We want $G \not\models \varphi(\vec{a}, b)$.
- Consider the subgroup $G_0 := \langle a_1, \ldots, a_n, b \rangle$ of G and set $H := G *_{\langle a_1, \ldots, a_n \rangle} G_0$.
- Let b' denote the copy of b in the "right factor".
- The map $a_i \mapsto a_i$, $b \mapsto b'$ is an isomorphism between subgroups of H and so in the corresponding **HNN extension** $H_1 \supseteq H$, there is $c \in H_1$ such that $c a_i c^{-1} = a_i$ and $c b c^{-1} = b' \neq b$.
- Consequently, $H_1 \models \neg\varphi(\vec{a}, b)$ whence $G \models \neg\varphi(\vec{a}, b)$ since G is e.c.

The proof of Macintyre's theorem

- $\varphi(\vec{x}, y) \equiv \forall z \left(\bigwedge_{i=1}^{n} [z, x_i] = e \rightarrow [z, y] = e \right).$
- It is clear that $G \models \varphi(\vec{a}, b)$ for any group G whenever $b \in \langle a_1, \ldots, a_n \rangle$.
- Now suppose that G is e.c. and $a_1, \ldots, a_n, b \in G$ are such that $b \notin \langle a_1, \ldots, a_n \rangle$. We want $G \not\models \varphi(\vec{a}, b)$.
- Consider the subgroup $G_0 := \langle a_1, \ldots, a_n, b \rangle$ of G and set $H := G *_{\langle a_1, \ldots, a_n \rangle} G_0$.
- Let b' denote the copy of b in the "right factor".
- The map $a_i \mapsto a_i$, $b \mapsto b'$ is an isomorphism between subgroups of H and so in the corresponding **HNN extension** $H_1 \supseteq H$, there is $c \in H_1$ such that $ca_ic^{-1} = a_i$ and $cbc^{-1} = b' \neq b$.
- Consequently, $H_1 \models \neg\varphi(\vec{a}, b)$ whence $G \models \neg\varphi(\vec{a}, b)$ since G is e.c.

The proof of Macintyre's theorem

- $\varphi(\vec{x}, y) \equiv \forall z \left(\bigwedge_{i=1}^{n} [z, x_i] = e \rightarrow [z, y] = e\right)$.
- It is clear that $G \models \varphi(\vec{a}, b)$ for any group G whenever $b \in \langle a_1, \ldots, a_n \rangle$.
- Now suppose that G is e.c. and $a_1, \ldots, a_n, b \in G$ are such that $b \notin \langle a_1, \ldots, a_n \rangle$. We want $G \not\models \varphi(\vec{a}, b)$.
- Consider the subgroup $G_0 := \langle a_1, \ldots, a_n, b \rangle$ of G and set $H := G *_{\langle a_1, \ldots, a_n \rangle} G_0$.
- Let b' denote the copy of b in the "right factor".
- The map $a_i \mapsto a_i$, $b \mapsto b'$ is an isomorphism between subgroups of H and so in the corresponding **HNN extension** $H_1 \supseteq H$, there is $c \in H_1$ such that $ca_ic^{-1} = a_i$ and $cbc^{-1} = b' \neq b$.
- Consequently, $H_1 \models \neg\varphi(\vec{a}, b)$ whence $G \models \neg\varphi(\vec{a}, b)$ since G is e.c.

The proof of Macintyre's theorem

- $\varphi(\vec{x}, y) \equiv \forall z \left(\bigwedge_{i=1}^{n} [z, x_i] = e \to [z, y] = e \right)$.
- It is clear that $G \models \varphi(\vec{a}, b)$ for any group G whenever $b \in \langle a_1, \ldots, a_n \rangle$.
- Now suppose that G is e.c. and $a_1, \ldots, a_n, b \in G$ are such that $b \notin \langle a_1, \ldots, a_n \rangle$. We want $G \not\models \varphi(\vec{a}, b)$.
- Consider the subgroup $G_0 := \langle a_1, \ldots, a_n, b \rangle$ of G and set $H := G *_{\langle a_1, \ldots, a_n \rangle} G_0$.
- Let b' denote the copy of b in the "right factor".
- The map $a_i \mapsto a_i$, $b \mapsto b'$ is an isomorphism between subgroups of H and so in the corresponding **HNN extension** $H_1 \supseteq H$, there is $c \in H_1$ such that $ca_ic^{-1} = a_i$ and $cbc^{-1} = b' \neq b$.
- Consequently, $H_1 \models \neg\varphi(\vec{a}, b)$ whence $G \models \neg\varphi(\vec{a}, b)$ since G is e.c.

A version for e.c. tracial von Neumann algebras?

- I had been studying e.c. tracial von Neumann algebras and was wondering if a version of this theorem would hold in that context.
- In continuous logic, “existential” statements use the quantifier $\inf$ which act like approximate existential quantifiers.
- Thus, we would have elements approximately commuting with some subalgebra and would hope that would mean there would be a “nearby” element which actually commutes with this subalgebra.
- This is a well-studied phenomena in operator algebras known as spectral gap.
- We then obtain an analog that is computability-theoretic and which concerns computing conditional expectations onto spectral gap subalgebras.

Defining von Neumann algebras

- H a complex Hilbert space, $B(H)$ the set of bounded operators on H.
- The **weak operator topology** (WOT) on $B(H)$ is induced by the family of semi-norms given by $a \mapsto |\langle a\zeta, \eta\rangle|$ for $\zeta, \eta \in H$.
- $M \subseteq B(H)$ is a **von Neumann algebra** if it is a unital *-algebra closed in the weak operator topology.
- von Neumann's famous **bicommutant theorem** states that this is equivalent to M being a unital *-algebra of $B(H)$ which satisfies $M = M''$, where, for any $X \subseteq B(H)$, we set

$$X' := \{a \in B(H) : ab = ba \text{ for all } b \in X\}.$$

Defining von Neumann algebras

- H a complex Hilbert space, $B(H)$ the set of bounded operators on H.
- The **weak operator topology** (WOT) on $B(H)$ is induced by the family of semi-norms given by $a \mapsto |\langle a\zeta, \eta\rangle|$ for $\zeta, \eta \in H$.
- $M \subseteq B(H)$ is a **von Neumann algebra** if it is a unital *-algebra closed in the weak operator topology.
- von Neumann's famous **bicommutant theorem** states that this is equivalent to M being a unital *-algebra of $B(H)$ which satisfies $M = M''$, where, for any $X \subseteq B(H)$, we set

$$X' := \{a \in B(H) : ab = ba \text{ for all } b \in X\}.$$

Defining von Neumann algebras

- H a complex Hilbert space, $B(H)$ the set of bounded operators on H.
- The **weak operator topology** (WOT) on $B(H)$ is induced by the family of semi-norms given by $a \mapsto |\langle a\zeta, \eta\rangle|$ for $\zeta, \eta \in H$.
- $M \subseteq B(H)$ is a **von Neumann algebra** if it is a unital *-algebra closed in the weak operator topology.
- von Neumann's famous **bicommutant theorem** states that this is equivalent to M being a unital *-algebra of $B(H)$ which satisfies $M = M''$, where, for any $X \subseteq B(H)$, we set

$$X' := \{a \in B(H) : ab = ba \text{ for all } b \in X\}.$$

Defining von Neumann algebras

- H a complex Hilbert space, $B(H)$ the set of bounded operators on H.
- The **weak operator topology** (WOT) on $B(H)$ is induced by the family of semi-norms given by $a \mapsto |\langle a\zeta, \eta\rangle|$ for $\zeta, \eta \in H$.
- $M \subseteq B(H)$ is a **von Neumann algebra** if it is a unital *-algebra closed in the weak operator topology.
- von Neumann's famous **bicommutant theorem** states that this is equivalent to M being a unital *-algebra of $B(H)$ which satisfies $M = M''$, where, for any $X \subseteq B(H)$, we set

$$X' := \{a \in B(H) : ab = ba \text{ for all } b \in X\}.$$

Traces and tracial von Neumann algebras

Definition

A linear functional τ on a von Neumann-algebra M is a (finite, normalized) **trace** if it is:

- **positive**: ($\tau(a^*a) \geq 0$ for all $a \in M$),
- **tracial**: $\tau(a^*a) = \tau(aa^*)$ for all $a \in M$,
- **normal**: WOT continuous on the operator norm unit ball, and
- $\tau(1) = 1$.

We say it is **faithful** if $\tau(a^*a) = 0$ implies $a = 0$.

A **tracial von Neumann algebra** is a pair (M,τ) consisting of a von Neumann algebra and a faithful trace τ on M.

τ induces an inner product $\langle a, b\rangle_\tau := \tau(b^*a)$ on M with corresponding norm $\|a\|_2 = \|a\|_\tau = \sqrt{\tau(a^*a)}$.

The completion is a Hilbert space denoted $L^2(M, \tau)$.

Traces and tracial von Neumann algebras

Definition

A linear functional τ on a von Neumann-algebra M is a (finite, normalized) **trace** if it is:

- **positive**: ($\tau(a^*a) \geq 0$ for all $a \in M$),
- **tracial**: $\tau(a^*a) = \tau(aa^*)$ for all $a \in M$,
- **normal**: WOT continuous on the operator norm unit ball, and
- $\tau(1) = 1$.

We say it is **faithful** if $\tau(a^*a) = 0$ implies $a = 0$.

A **tracial von Neumann algebra** is a pair (M,τ) consisting of a von Neumann algebra and a faithful trace τ on M.

τ induces an inner product $\langle a, b\rangle_\tau := \tau(b^*a)$ on M with corresponding norm $\|a\|_2 = \|a\|_\tau = \sqrt{\tau(a^*a)}$.

The completion is a Hilbert space denoted $L^2(M,\tau)$.

Traces and tracial von Neumann algebras

Definition

A linear functional τ on a von Neumann-algebra M is a (finite, normalized) **trace** if it is:

- **positive**: ($\tau(a^*a) \geq 0$ for all $a \in M$),
- **tracial**: $\tau(a^*a) = \tau(aa^*)$ for all $a \in M$,
- **normal**: WOT continuous on the operator norm unit ball, and
- $\tau(1) = 1$.

We say it is **faithful** if $\tau(a^*a) = 0$ implies $a = 0$.

A **tracial von Neumann algebra** is a pair (M,τ) consisting of a von Neumann algebra and a faithful trace τ on M.
τ induces an inner product $\langle a, b\rangle_\tau := \tau(b^*a)$ on M with corresponding norm $\|a\|_2 = \|a\|_\tau = \sqrt{\tau(a^*a)}$.
The completion is a Hilbert space denoted $L^2(M, \tau)$.

Traces and tracial von Neumann algebras

Definition

A linear functional τ on a von Neumann-algebra M is a (finite, normalized) **trace** if it is:

- **positive**: ($\tau(a^*a) \geq 0$ for all $a \in M$),
- **tracial**: $\tau(a^*a) = \tau(aa^*)$ for all $a \in M$,
- **normal**: WOT continuous on the operator norm unit ball, and
- $\tau(1) = 1$.

We say it is **faithful** if $\tau(a^*a) = 0$ implies $a = 0$.

A **tracial von Neumann algebra** is a pair (M,τ) consisting of a von Neumann algebra and a faithful trace τ on M.
τ induces an inner product $\langle a, b\rangle_\tau := \tau(b^*a)$ on M with corresponding norm $\|a\|_2 = \|a\|_\tau = \sqrt{\tau(a^*a)}$.
The completion is a Hilbert space denoted $L^2(M,\tau)$.

II_1 factors

Definition

If M is a von Neumann algebra, its **center** is

$$Z(M) = M' \cap M := \{a \in M \ : \ ab = ba \text{ for all } b \in M\}.$$

M is called a **factor** if $Z(M)$ is trivial, that is, $Z(M) = \mathbb{C} \cdot 1$.
A $\mathbf{II}_1$ **factor** is an infinite-dimensional factor that admits a trace.

- Factors are the "irreducible" von Neumann algebras: every von Neumann algebra is a direct integral of factors.
- The trace on a II_1 factor is unique.

II_1 factors

Definition

If M is a von Neumann algebra, its **center** is

$$Z(M) = M' \cap M := \{a \in M \ : \ ab = ba \text{ for all } b \in M\}.$$

M is called a **factor** if $Z(M)$ is trivial, that is, $Z(M) = \mathbb{C} \cdot 1$.
A **II_1 factor** is an infinite-dimensional factor that admits a trace.

- Factors are the “irreducible” von Neumann algebras: every von Neumann algebra is a direct integral of factors.
- The trace on a II_1 factor is unique.

Examples

- $M_n(\mathbb{C})$ with the normalized trace is a tracial factor, but not a II_1 factor; $B(H)$ for infinite-dimensional H is a factor but is not tracial.
- The abelian von Neumann algebras are all of the form $L^\infty(X,\mu)$, viewed as multiplication operators on $L^2(X,\mu)$. $\tau(f) := \int_X f d\mu$ is a trace on $L^\infty(X,\mu)$ whose completion is (of course) $L^2(X,\mu)$. They are the complete opposite of factors.
- If G is a group, then for $g \in G$, let $u_g : G \to B(\ell^2(G))$ be the unitary operator determined by

  $$u_g(\zeta_h) = \zeta_{gh}.$$

  $L(G)$ is the von Neumann algebra generated by the u_g's, called the **group von Neumann algebra of** G.
  It is tracial: for $a \in L(G)$, let $\tau(a) = \langle a(\zeta_e), \zeta_e \rangle$. The completion with respect to this norm is $\ell^2(G)$.
  It is a II_1 factor when G is **ICC** (all nontrivial conjugacy classes are infinite).

Examples

- $M_n(\mathbb{C})$ with the normalized trace is a tracial factor, but not a II_1 factor; $B(H)$ for infinite-dimensional H is a factor but is not tracial.
- The abelian von Neumann algebras are all of the form $L^\infty(X, \mu)$, viewed as multiplication operators on $L^2(X, \mu)$. $\tau(f) := \int_X f d\mu$ is a trace on $L^\infty(X, \mu)$ whose completion is (of course) $L^2(X, \mu)$. They are the complete opposite of factors.
- If G is a group, then for $g \in G$, let $u_g : G \to B(\ell^2(G))$ be the unitary operator determined by

$$u_g(\zeta_h) = \zeta_{gh}.$$

$L(G)$ is the von Neumann algebra generated by the u_g's, called the **group von Neumann algebra of** G.
It is tracial: for $a \in L(G)$, let $\tau(a) = \langle a(\zeta_e), \zeta_e \rangle$. The completion with respect to this norm is $\ell^2(G)$.
It is a II_1 factor when G is **ICC** (all nontrivial conjugacy classes are infinite).

Examples

- $M_n(\mathbb{C})$ with the normalized trace is a tracial factor, but not a II_1 factor; $B(H)$ for infinite-dimensional H is a factor but is not tracial.
- The abelian von Neumann algebras are all of the form $L^\infty(X, \mu)$, viewed as multiplication operators on $L^2(X, \mu)$. $\tau(f) := \int_X f d\mu$ is a trace on $L^\infty(X, \mu)$ whose completion is (of course) $L^2(X, \mu)$. They are the complete opposite of factors.
- If G is a group, then for $g \in G$, let $u_g : G \to B(\ell^2(G))$ be the unitary operator determined by

$$u_g(\zeta_h) = \zeta_{gh}.$$

$L(G)$ is the von Neumann algebra generated by the u_g's, called the **group von Neumann algebra of** G.
It is tracial: for $a \in L(G)$, let $\tau(a) = \langle a(\zeta_e), \zeta_e \rangle$. The completion with respect to this norm is $\ell^2(G)$.
It is a II_1 factor when G is **ICC** (all nontrivial conjugacy classes are infinite).

Examples

- $M_n(\mathbb{C})$ with the normalized trace is a tracial factor, but not a II_1 factor; $B(H)$ for infinite-dimensional H is a factor but is not tracial.
- The abelian von Neumann algebras are all of the form $L^\infty(X, \mu)$, viewed as multiplication operators on $L^2(X, \mu)$. $\tau(f) := \int_X f d\mu$ is a trace on $L^\infty(X, \mu)$ whose completion is (of course) $L^2(X, \mu)$. They are the complete opposite of factors.
- If G is a group, then for $g \in G$, let $u_g : G \to B(\ell^2(G))$ be the unitary operator determined by

$$u_g(\zeta_h) = \zeta_{gh}.$$

$L(G)$ is the von Neumann algebra generated by the u_g's, called the **group von Neumann algebra of** G.

It is tracial: for $a \in L(G)$, let $\tau(a) = \langle a(\zeta_e), \zeta_e \rangle$. The completion with respect to this norm is $\ell^2(G)$.

It is a II_1 factor when G is **ICC** (all nontrivial conjugacy classes are infinite).

Examples

- $M_n(\mathbb{C})$ with the normalized trace is a tracial factor, but not a II_1 factor; $B(H)$ for infinite-dimensional H is a factor but is not tracial.
- The abelian von Neumann algebras are all of the form $L^\infty(X,\mu)$, viewed as multiplication operators on $L^2(X,\mu)$. $\tau(f) := \int_X f d\mu$ is a trace on $L^\infty(X,\mu)$ whose completion is (of course) $L^2(X,\mu)$. They are the complete opposite of factors.
- If G is a group, then for $g \in G$, let $u_g : G \to B(\ell^2(G))$ be the unitary operator determined by

$$u_g(\zeta_h) = \zeta_{gh}.$$

$L(G)$ is the von Neumann algebra generated by the u_g's, called the **group von Neumann algebra of** G.
It is tracial: for $a \in L(G)$, let $\tau(a) = \langle a(\zeta_e), \zeta_e \rangle$. The completion with respect to this norm is $\ell^2(G)$.
It is a II_1 factor when G is **ICC** (all nontrivial conjugacy classes are infinite).

Examples

- $M_n(\mathbb{C})$ with the normalized trace is a tracial factor, but not a II_1 factor; $B(H)$ for infinite-dimensional H is a factor but is not tracial.
- The abelian von Neumann algebras are all of the form $L^\infty(X,\mu)$, viewed as multiplication operators on $L^2(X,\mu)$. $\tau(f) := \int_X f d\mu$ is a trace on $L^\infty(X,\mu)$ whose completion is (of course) $L^2(X,\mu)$. They are the complete opposite of factors.
- If G is a group, then for $g \in G$, let $u_g : G \to B(\ell^2(G))$ be the unitary operator determined by

  $$u_g(\zeta_h) = \zeta_{gh}.$$

  $L(G)$ is the von Neumann algebra generated by the u_g's, called the **group von Neumann algebra of** G.
  It is tracial: for $a \in L(G)$, let $\tau(a) = \langle a(\zeta_e), \zeta_e \rangle$. The completion with respect to this norm is $\ell^2(G)$.
  It is a II_1 factor when G is **ICC** (all nontrivial conjugacy classes are infinite).

Some syntax

- An **atomic formula** is one of the form $\Re \mathtt{tr}(p(x))$ or $\Im \mathtt{tr}(p(x))$ for some *-polynomial $p(x)$.
- We obtain the class of all **formulae** by closing under the following two operations:
  - If $\varphi_1, \ldots, \varphi_n$ are formulae and $f : \mathbb{R}^n \to \mathbb{R}$ is continuous, then $f(\varphi_1, \ldots, \varphi_n)$ is also a formula.
  - If φ is a formula and x is a variable, then for every n, $\inf_{\|x\| \leq n} \varphi$ and $\sup_{\|x\| \leq n} \varphi$ are formulae.
- If $\varphi(x)$ is a formula with **free variables** $x = (x_1, \ldots, x_n)$ and M is a tracial von Neumann algebra, we get a natural **interpretation** function $\varphi^M : M^n \to \mathbb{R}$.
- A **sentence** is a formula without free variables. If σ is a sentence and M is a tracial von Neumann algebra, then $\sigma^M \in \mathbb{R}$.
- For example, $\sigma := \inf_{\|x\| \leq 1} \max(\|x - x^*\|, \|x - x^2\|, |\operatorname{tr}(x) - \frac{1}{\pi}|)$ is a sentence and $\sigma^M = 0$ if and only if M has a projection of trace $\frac{1}{\pi}$.

Some syntax

- An **atomic formula** is one of the form $\Re\,\mathrm{tr}(p(x))$ or $\Im\,\mathrm{tr}(p(x))$ for some *-polynomial $p(x)$.
- We obtain the class of all **formulae** by closing under the following two operations:
  - If $\varphi_1, \ldots, \varphi_n$ are formulae and $f : \mathbb{R}^n \to \mathbb{R}$ is continuous, then $f(\varphi_1, \ldots, \varphi_n)$ is also a formula.
  - If φ is a formula and x is a variable, then for every n, $\inf_{\|x\| \leq n} \varphi$ and $\sup_{\|x\| \leq n} \varphi$ are formulae.
- If $\varphi(x)$ is a formula with **free variables** $x = (x_1, \ldots, x_n)$ and M is a tracial von Neumann algebra, we get a natural **interpretation** function $\varphi^M : M^n \to \mathbb{R}$.
- A **sentence** is a formula without free variables. If σ is a sentence and M is a tracial von Neumann algebra, then $\sigma^M \in \mathbb{R}$.
- For example, $\sigma := \inf_{\|x\| \leq 1} \max(\|x - x^*\|, \|x - x^2\|, |\mathrm{tr}(x) - \frac{1}{\pi}|)$ is a sentence and $\sigma^M = 0$ if and only if M has a projection of trace $\frac{1}{\pi}$.

Some syntax

- An **atomic formula** is one of the form $\Re\,\mathrm{tr}(p(x))$ or $\Im\,\mathrm{tr}(p(x))$ for some *-polynomial $p(x)$.
- We obtain the class of all **formulae** by closing under the following two operations:
  - If $\varphi_1, \ldots, \varphi_n$ are formulae and $f : \mathbb{R}^n \to \mathbb{R}$ is continuous, then $f(\varphi_1, \ldots, \varphi_n)$ is also a formula.
  - If φ is a formula and x is a variable, then for every n, $\inf_{\|x\| \leq n} \varphi$ and $\sup_{\|x\| \leq n} \varphi$ are formulae.
- If $\varphi(x)$ is a formula with **free variables** $x = (x_1, \ldots, x_n)$ and M is a tracial von Neumann algebra, we get a natural **interpretation** function $\varphi^M : M^n \to \mathbb{R}$.
- A **sentence** is a formula without free variables. If σ is a sentence and M is a tracial von Neumann algebra, then $\sigma^M \in \mathbb{R}$.
- For example, $\sigma := \inf_{\|x\| \leq 1} \max(\|x - x^*\|, \|x - x^2\|, |\,\mathrm{tr}(x) - \frac{1}{\pi}|)$ is a sentence and $\sigma^M = 0$ if and only if M has a projection of trace $\frac{1}{\pi}$.

Some syntax

- An **atomic formula** is one of the form $\Re\,\mathrm{tr}(p(x))$ or $\Im\,\mathrm{tr}(p(x))$ for some *-polynomial $p(x)$.
- We obtain the class of all **formulae** by closing under the following two operations:
  - If $\varphi_1, \ldots, \varphi_n$ are formulae and $f : \mathbb{R}^n \to \mathbb{R}$ is continuous, then $f(\varphi_1, \ldots, \varphi_n)$ is also a formula.
  - If φ is a formula and x is a variable, then for every n, $\inf_{\|x\|\leq n} \varphi$ and $\sup_{\|x\|\leq n} \varphi$ are formulae.
- If $\varphi(x)$ is a formula with **free variables** $x = (x_1, \ldots, x_n)$ and M is a tracial von Neumann algebra, we get a natural **interpretation** function $\varphi^M : M^n \to \mathbb{R}$.
- A **sentence** is a formula without free variables. If σ is a sentence and M is a tracial von Neumann algebra, then $\sigma^M \in \mathbb{R}$.
- For example, $\sigma := \inf_{\|x\|\leq 1} \max(\|x - x^*\|, \|x - x^2\|, |\mathrm{tr}(x) - \frac{1}{\pi}|)$ is a sentence and $\sigma^M = 0$ if and only if M has a projection of trace $\frac{1}{\pi}$.

Some syntax

- An **atomic formula** is one of the form $\Re\,\mathrm{tr}(p(x))$ or $\Im\,\mathrm{tr}(p(x))$ for some *-polynomial $p(x)$.
- We obtain the class of all **formulae** by closing under the following two operations:
  - If $\varphi_1, \ldots, \varphi_n$ are formulae and $f : \mathbb{R}^n \to \mathbb{R}$ is continuous, then $f(\varphi_1, \ldots, \varphi_n)$ is also a formula.
  - If φ is a formula and x is a variable, then for every n, $\inf_{\|x\|\leq n} \varphi$ and $\sup_{\|x\|\leq n} \varphi$ are formulae.
- If $\varphi(x)$ is a formula with **free variables** $x = (x_1, \ldots, x_n)$ and M is a tracial von Neumann algebra, we get a natural **interpretation** function $\varphi^M : M^n \to \mathbb{R}$.
- A **sentence** is a formula without free variables. If σ is a sentence and M is a tracial von Neumann algebra, then $\sigma^M \in \mathbb{R}$.
- For example, $\sigma := \inf_{\|x\|\leq 1} \max(\|x - x^*\|, \|x - x^2\|, |\,\mathrm{tr}(x) - \frac{1}{\pi}|)$ is a sentence and $\sigma^M = 0$ if and only if M has a projection of trace $\frac{1}{\pi}$.

Some syntax

- An **atomic formula** is one of the form $\Re\operatorname{tr}(p(x))$ or $\Im\operatorname{tr}(p(x))$ for some *-polynomial $p(x)$.
- We obtain the class of all **formulae** by closing under the following two operations:
  - If $\varphi_1, \ldots, \varphi_n$ are formulae and $f : \mathbb{R}^n \to \mathbb{R}$ is continuous, then $f(\varphi_1, \ldots, \varphi_n)$ is also a formula.
  - If φ is a formula and x is a variable, then for every n, $\inf_{\|x\| \leq n} \varphi$ and $\sup_{\|x\| \leq n} \varphi$ are formulae.
- If $\varphi(x)$ is a formula with **free variables** $x = (x_1, \ldots, x_n)$ and M is a tracial von Neumann algebra, we get a natural **interpretation** function $\varphi^M : M^n \to \mathbb{R}$.
- A **sentence** is a formula without free variables. If σ is a sentence and M is a tracial von Neumann algebra, then $\sigma^M \in \mathbb{R}$.
- For example, $\sigma := \inf_{\|x\| \leq 1} \max(\|x - x^*\|, \|x - x^2\|, |\operatorname{tr}(x) - \frac{1}{\pi}|)$ is a sentence and $\sigma^M = 0$ if and only if M has a projection of trace $\frac{1}{\pi}$.

Existentially closed tracial von Neumann algebras

Definition

A tracial von Neumann algebra M is called **existentially closed (e.c.)** if: for all quantifier-free formulae $\varphi(\vec{x}, \vec{y})$, all $\vec{a} \in M$, and all $N \supseteq M$, we have

$$\left(\inf_{\vec{y}} \varphi(\vec{a}, \vec{y})\right)^M = \left(\inf_{\vec{y}} \varphi(\vec{a}, \vec{y})\right)^N.$$

We know many things about e.c. tracial von Neumann algebras:

- They exist! In fact, every tracial von Neumann algebra is contained in an e.c. one.
- E.c. tracial von Neumann algebras must be II_1 factors.
- We know other von Neumann algebraic properties of them: they are McDuff II_1 factors with only approximately inner automorphisms...
- Like groups, we do not know any "concrete" examples.

Existentially closed tracial von Neumann algebras

Definition

A tracial von Neumann algebra M is called **existentially closed (e.c.)** if: for all quantifier-free formulae $\varphi(\vec{x}, \vec{y})$, all $\vec{a} \in M$, and all $N \supseteq M$, we have

$$\left(\inf_{\vec{y}} \varphi(\vec{a}, \vec{y})\right)^M = \left(\inf_{\vec{y}} \varphi(\vec{a}, \vec{y})\right)^N.$$

We know many things about e.c. tracial von Neumann algebras:

- They exist! In fact, every tracial von Neumann algebra is contained in an e.c. one.
- E.c. tracial von Neumann algebras must be II_1 factors.
- We know other von Neumann algebraic properties of them: they are McDuff II_1 factors with only approximately inner automorphisms...
- Like groups, we do not know any "concrete" examples.

Existentially closed tracial von Neumann algebras

Definition

A tracial von Neumann algebra M is called **existentially closed (e.c.)** if: for all quantifier-free formulae $\varphi(\vec{x}, \vec{y})$, all $\vec{a} \in M$, and all $N \supseteq M$, we have

$$\left(\inf_{\vec{y}} \varphi(\vec{a}, \vec{y})\right)^M = \left(\inf_{\vec{y}} \varphi(\vec{a}, \vec{y})\right)^N.$$

We know many things about e.c. tracial von Neumann algebras:

- They exist! In fact, every tracial von Neumann algebra is contained in an e.c. one.
- E.c. tracial von Neumann algebras must be II_1 factors.
- We know other von Neumann algebraic properties of them: they are McDuff II_1 factors with only approximately inner automorphisms...
- Like groups, we do not know any "concrete" examples.

Existentially closed tracial von Neumann algebras

Definition

A tracial von Neumann algebra M is called **existentially closed (e.c.)** if: for all quantifier-free formulae $\varphi(\vec{x}, \vec{y})$, all $\vec{a} \in M$, and all $N \supseteq M$, we have

$$\left(\inf_{\vec{y}} \varphi(\vec{a}, \vec{y})\right)^M = \left(\inf_{\vec{y}} \varphi(\vec{a}, \vec{y})\right)^N.$$

We know many things about e.c. tracial von Neumann algebras:

- They exist! In fact, every tracial von Neumann algebra is contained in an e.c. one.
- E.c. tracial von Neumann algebras must be II_1 factors.
- We know other von Neumann algebraic properties of them: they are McDuff II_1 factors with only approximately inner automorphisms...
- Like groups, we do not know any "concrete" examples.

Existentially closed tracial von Neumann algebras

Definition

A tracial von Neumann algebra M is called **existentially closed (e.c.)** if: for all quantifier-free formulae $\varphi(\vec{x}, \vec{y})$, all $\vec{a} \in M$, and all $N \supseteq M$, we have

$$\left(\inf_{\vec{y}} \varphi(\vec{a}, \vec{y})\right)^M = \left(\inf_{\vec{y}} \varphi(\vec{a}, \vec{y})\right)^N.$$

We know many things about e.c. tracial von Neumann algebras:

- They exist! In fact, every tracial von Neumann algebra is contained in an e.c. one.
- E.c. tracial von Neumann algebras must be II_1 factors.
- We know other von Neumann algebraic properties of them: they are McDuff II_1 factors with only approximately inner automorphisms...
- Like groups, we do not know any "concrete" examples.

Existentially closed tracial von Neumann algebras

Definition

A tracial von Neumann algebra M is called **existentially closed (e.c.)** if: for all quantifier-free formulae $\varphi(\vec{x}, \vec{y})$, all $\vec{a} \in M$, and all $N \supseteq M$, we have

$$\left(\inf_{\vec{y}} \varphi(\vec{a}, \vec{y})\right)^M = \left(\inf_{\vec{y}} \varphi(\vec{a}, \vec{y})\right)^N.$$

We know many things about e.c. tracial von Neumann algebras:

- They exist! In fact, every tracial von Neumann algebra is contained in an e.c. one.
- E.c. tracial von Neumann algebras must be II_1 factors.
- We know other von Neumann algebraic properties of them: they are McDuff II_1 factors with only approximately inner automorphisms...
- Like groups, we do not know any "concrete" examples.

Computable structure theory

Definition

Throughout, M_1 denotes the operator norm unit ball.

1. Given $A \subseteq M_1$, we let $\langle A \rangle$ be the smallest subset of M_1 containing A and closed under rational rounded combinations, multiplication, and adjoint.
2. We say that A **generates** M if $\langle A \rangle$ is 2-norm dense in M_1.
3. A **presentation** of M is a pair $M^\# := (M, (a_n)_{n \in \mathbb{N}})$, where $\{a_n : n \in \mathbb{N}\} \subseteq M_1$ generates M.
   - Elements of the sequence $(a_n)_{n \in \mathbb{N}}$ are called **special points** of the presentation.
   - Elements of $\langle \{a_n : n \in \mathbb{N}\} \rangle$ are called **rational points** of the presentation.

Computable structure theory

Definition

Throughout, M_1 denotes the operator norm unit ball.

1. Given $A \subseteq M_1$, we let $\langle A \rangle$ be the smallest subset of M_1 containing A and closed under rational rounded combinations, multiplication, and adjoint.
2. We say that A **generates** M if $\langle A \rangle$ is 2-norm dense in M_1.
3. A **presentation** of M is a pair $M^{\#} := (M, (a_n)_{n \in \mathbb{N}})$, where $\{a_n : n \in \mathbb{N}\} \subseteq M_1$ generates M.
   - Elements of the sequence $(a_n)_{n \in \mathbb{N}}$ are called **special points** of the presentation.
   - Elements of $\langle \{a_n : n \in \mathbb{N}\} \rangle$ are called **rational points** of the presentation.

Computable structure theory

Definition

Throughout, M_1 denotes the operator norm unit ball.

1. Given $A \subseteq M_1$, we let $\langle A \rangle$ be the smallest subset of M_1 containing A and closed under rational rounded combinations, multiplication, and adjoint.
2. We say that A **generates** M if $\langle A \rangle$ is 2-norm dense in M_1.
3. A **presentation** of M is a pair $M^\# := (M, (a_n)_{n \in \mathbb{N}})$, where $\{a_n : n \in \mathbb{N}\} \subseteq M_1$ generates M.
   - Elements of the sequence $(a_n)_{n \in \mathbb{N}}$ are called **special points** of the presentation.
   - Elements of $\langle \{a_n : n \in \mathbb{N}\} \rangle$ are called **rational points** of the presentation.

Computable structure theory

Definition

Throughout, M_1 denotes the operator norm unit ball.

1. Given $A \subseteq M_1$, we let $\langle A \rangle$ be the smallest subset of M_1 containing A and closed under rational rounded combinations, multiplication, and adjoint.
2. We say that A **generates** M if $\langle A \rangle$ is 2-norm dense in M_1.
3. A **presentation** of M is a pair $M^\# := (M, (a_n)_{n\in\mathbb{N}})$, where $\{a_n : n \in \mathbb{N}\} \subseteq M_1$ generates M.
   - Elements of the sequence $(a_n)_{n\in\mathbb{N}}$ are called **special points** of the presentation.
   - Elements of $\langle \{a_n : n \in \mathbb{N}\} \rangle$ are called **rational points** of the presentation.

More computable structure theory

Remark

Given a presentation $M^\#$ of M, it is possible to computably enumerate the rational points of $M^\#$. Consequently, it makes sense to consider algorithms which take rational points of $M^\#$ as inputs and/or outputs.

Definition

Suppose $M^\#$ is a presentation of M and **D** is an oracle.

1. $x \in M_1$ is a **D-computable point of** $M^\#$ if there is a **D**-algorithm such that, upon input $k \in \mathbb{N}$, returns a rational point $p \in M^\#$ with $d(x, p) < 2^{-k}$.
2. $M^\#$ is a **D-computable presentation** if there is a **D**-algorithm such that, upon input rational point $p \in M^\#$ and $k \in \mathbb{N}$, returns a rational number q such that $|\|p\|_2 - q| < 2^{-k}$.

More computable structure theory

Remark

Given a presentation $M^\#$ of M, it is possible to computably enumerate the rational points of $M^\#$. Consequently, it makes sense to consider algorithms which take rational points of $M^\#$ as inputs and/or outputs.

Definition

Suppose $M^\#$ is a presentation of M and **D** is an oracle.

1. $x \in M_1$ is a **D-computable point of** $M^\#$ if there is a **D**-algorithm such that, upon input $k \in \mathbb{N}$, returns a rational point $p \in M^\#$ with $d(x, p) < 2^{-k}$.
2. $M^\#$ is a **D-computable presentation** if there is a **D**-algorithm such that, upon input rational point $p \in M^\#$ and $k \in \mathbb{N}$, returns a rational number q such that $|\|p\|_2 - q| < 2^{-k}$.

More computable structure theory

Remark

Given a presentation $M^\#$ of M, it is possible to computably enumerate the rational points of $M^\#$. Consequently, it makes sense to consider algorithms which take rational points of $M^\#$ as inputs and/or outputs.

Definition

Suppose $M^\#$ is a presentation of M and **D** is an oracle.

1. $x \in M_1$ is a **D-computable point of $M^\#$** if there is a **D**-algorithm such that, upon input $k \in \mathbb{N}$, returns a rational point $p \in M^\#$ with $d(x, p) < 2^{-k}$.
2. $M^\#$ is a **D-computable presentation** if there is a **D**-algorithm such that, upon input rational point $p \in M^\#$ and $k \in \mathbb{N}$, returns a rational number q such that $|\|p\|_2 - q| < 2^{-k}$.

Computable maps

Definition

Suppose that M and N are tracial von Neumann algebras with presentations $M^{\#}$ and $N^{\dagger}$ respectively. Further suppose that $f : M^m \to N$ is a Lipshitz map and **D** is an oracle. Then f is a **D-computable map from** $M^{\#}$ **into** $N^{\dagger}$ if there is a **D**-algorithm such that, upon input a tuple of rational points $\vec{p} \in (M^{\#})^m$ and $k \in \mathbb{N}$, returns a rational point $p' \in N^{\dagger}$ such that $d(f(\vec{p}), p') < 2^{-k}$.

Conditional expectations

- Recall from classical probability theory that if $(X, \mathcal{B}, \mu)$ is a probability space and $\mathcal{C}$ is a sub-σ-algebra of $\mathcal{B}$, then there is a **conditional expectation map** $E : L^2(X, \mathcal{B}, \mu) \to L^2(X, \mathcal{C}, \mu)$ given by orthogonal projection.
- In other words, $\int_A f d\mu = \int_A E(f) d\mu$ for all $A \in \mathcal{C}$.
- Similarly, if N is a subalgebra of the tracial von Neumann algebra (M, τ), then we can consider the orthogonal projection $E_N : L^2(M, \tau) \to L^2(N, \tau|N)$.
- Fact: $E_N(M) \subseteq N$. We call the restricted map $E_N : M \to N$ the **conditional expectation onto** N. (It is not a $*$-homomorphism, but it is a so-called ucp map.)

Conditional expectations

- Recall from classical probability theory that if $(X, \mathcal{B}, \mu)$ is a probability space and $\mathcal{C}$ is a sub-σ-algebra of $\mathcal{B}$, then there is a **conditional expectation map** $E : L^2(X, \mathcal{B}, \mu) \to L^2(X, \mathcal{C}, \mu)$ given by orthogonal projection.
- In other words, $\int_A f d\mu = \int_A E(f) d\mu$ for all $A \in \mathcal{C}$.
- Similarly, if N is a subalgebra of the tracial von Neumann algebra (M, τ), then we can consider the orthogonal projection $E_N : L^2(M, \tau) \to L^2(N, \tau|N)$.
- Fact: $E_N(M) \subseteq N$. We call the restricted map $E_N : M \to N$ the **conditional expectation onto** N. (It is not a $*$-homomorphism, but it is a so-called ucp map.)

Conditional expectations

- Recall from classical probability theory that if $(X, \mathcal{B}, \mu)$ is a probability space and $\mathcal{C}$ is a sub-σ-algebra of $\mathcal{B}$, then there is a **conditional expectation map** $E : L^2(X, \mathcal{B}, \mu) \to L^2(X, \mathcal{C}, \mu)$ given by orthogonal projection.
- In other words, $\int_A f d\mu = \int_A E(f) d\mu$ for all $A \in \mathcal{C}$.
- Similarly, if N is a subalgebra of the tracial von Neumann algebra (M, τ), then we can consider the orthogonal projection $E_N : L^2(M, \tau) \to L^2(N, \tau|N)$.
- Fact: $E_N(M) \subseteq N$. We call the restricted map $E_N : M \to N$ the **conditional expectation onto** N. (It is not a $*$-homomorphism, but it is a so-called ucp map.)

Conditional expectations

- Recall from classical probability theory that if $(X, \mathcal{B}, \mu)$ is a probability space and $\mathcal{C}$ is a sub-σ-algebra of $\mathcal{B}$, then there is a **conditional expectation map** $E : L^2(X, \mathcal{B}, \mu) \to L^2(X, \mathcal{C}, \mu)$ given by orthogonal projection.
- In other words, $\int_A f d\mu = \int_A E(f) d\mu$ for all $A \in \mathcal{C}$.
- Similarly, if N is a subalgebra of the tracial von Neumann algebra (M, τ), then we can consider the orthogonal projection $E_N : L^2(M, \tau) \to L^2(N, \tau|N)$.
- Fact: $E_N(M) \subseteq N$. We call the restricted map $E_N : M \to N$ the **conditional expectation onto** N. (It is not a $*$-homomorphism, but it is a so-called ucp map.)

Computability of conditional expectations

Definition

Suppose that N is a subalgebra of M and that $M^{\#}$ and $N^{\dagger}$ are presentations of M and N respectively. For an oracle **D**, we say that $(M^{\#}, N^{\dagger})$ is a **D-computable pair** if $M^{\#}$ is a **D**-computable presentation of M and the inclusion map $i : N^{\dagger} \to M^{\#}$ is a **D**-computable map. (It follows then that $N^{\dagger}$ is a **D**-computable presentation of N.)

Lemma

*Suppose that $(M^{\#}, N^{\dagger})$ is a **D**-computable pair. Then $E_N : M^{\#} \to N^{\dagger}$ is **D**-computable if and only if there is a **D**-algorithm which, upon input rational point $p \in M^{\#}$ and $k \in \mathbb{N}$, produces a rational number q such that $|d(p, N) - q| < 2^{-k}$.*

Computability of conditional expectations

Definition

Suppose that N is a subalgebra of M and that $M^{\#}$ and $N^{\dagger}$ are presentations of M and N respectively. For an oracle **D**, we say that $(M^{\#}, N^{\dagger})$ is a **D-computable pair** if $M^{\#}$ is a **D**-computable presentation of M and the inclusion map $i : N^{\dagger} \to M^{\#}$ is a **D**-computable map. (It follows then that $N^{\dagger}$ is a **D**-computable presentation of N.)

Lemma

Suppose that $(M^{\#}, N^{\dagger})$ is a **D***-computable pair. Then $E_N : M^{\#} \to N^{\dagger}$ is* **D***-computable if and only if there is a* **D***-algorithm which, upon input rational point $p \in M^{\#}$ and $k \in \mathbb{N}$, produces a rational number q such that $|d(p, N) - q| < 2^{-k}$.*

w-spectral gap and property (T)

Definition

N has **w-spectral gap in** M if, for any $\epsilon > 0$, there is a finite $F \subseteq N$ and $\delta > 0$ such that, for all $x \in M_1$, we have: if $\max_{y \in F} \|[x, y]\|_2 < \delta$, then there is $x' \in N' \cap M$ such that $d(x, x') < \epsilon$.

"Definition" (really a recent theorem of H. Tan)

The II_1 factor N has **property (T)** if it has w-spectral gap in any extension.

Example

If G is a countable discrete group, then $L(G)$ has property (T) if and only if G has property (T), e.g. $G = \mathrm{SL}_n(\mathbb{Z})$ for $n \geq 3$.

w-spectral gap and property (T)

Definition

N has **w-spectral gap in** M if, for any $\epsilon > 0$, there is a finite $F \subseteq N$ and $\delta > 0$ such that, for all $x \in M_1$, we have: if $\max_{y \in F} \|[x, y]\|_2 < \delta$, then there is $x' \in N' \cap M$ such that $d(x, x') < \epsilon$.

"Definition" (really a recent theorem of H. Tan)

The II_1 factor N has **property (T)** if it has w-spectral gap in any extension.

Example

If G is a countable discrete group, then $L(G)$ has property (T) if and only if G has property (T), e.g. $G = \mathrm{SL}_n(\mathbb{Z})$ for $n \geq 3$.

w-spectral gap and property (T)

Definition

N has **w-spectral gap in** M if, for any $\epsilon > 0$, there is a finite $F \subseteq N$ and $\delta > 0$ such that, for all $x \in M_1$, we have: if $\max_{y \in F} \|[x, y]\|_2 < \delta$, then there is $x' \in N' \cap M$ such that $d(x, x') < \epsilon$.

"Definition" (really a recent theorem of H. Tan)

The II_1 factor N has **property (T)** if it has w-spectral gap in any extension.

Example

If G is a countable discrete group, then $L(G)$ has property (T) if and only if G has property (T), e.g. $G = \mathrm{SL}_n(\mathbb{Z})$ for $n \geq 3$.

w-spectral gap functions

Definition

Suppose that N is a w-spectral gap subfactor of M and that $M^{\#}$ and $N^{\dagger} = (N; (a_n)_{n \in \mathbb{N}})$ are presentations of M and N respectively. We say that $f : \mathbb{N} \to \mathbb{N}$ is a **spectral gap function for** $(M^{\#}, N^{\dagger})$ if: for any $n \in \mathbb{N}$ and rational point $p \in M^{\#}$, if $\max_{1 \leq i \leq f(n)} \|[p, a_i]\|_2 < 2^{-f(n)}$, then $d(p, N' \cap M) < 2^{-n}$.

Remark

If N has w-spectral gap in M, then for any presentations $M^{\#}$ and $N^{\dagger}$ of M and N respectively, there is a spectral gap function f for $(M^{\#}, N^{\dagger})$.

w-spectral gap functions

Definition

Suppose that N is a w-spectral gap subfactor of M and that $M^{\#}$ and $N^{\dagger} = (N; (a_n)_{n\in\mathbb{N}})$ are presentations of M and N respectively. We say that $f : \mathbb{N} \to \mathbb{N}$ is a **spectral gap function for** $(M^{\#}, N^{\dagger})$ if: for any $n \in \mathbb{N}$ and rational point $p \in M^{\#}$, if $\max_{1\leq i\leq f(n)} \|[p, a_i]\|_2 < 2^{-f(n)}$, then $d(p, N' \cap M) < 2^{-n}$.

Remark

If N has w-spectral gap in M, then for any presentations $M^{\#}$ and $N^{\dagger}$ of M and N respectively, there is a spectral gap function f for $(M^{\#}, N^{\dagger})$.

Kazhdan presentations

Fact (Connes and Jones)

Suppose that N is a II_1 factor with property (T). Then there is $\epsilon > 0$, finite $F \subseteq N$, and $K > 0$ such that, for any $\delta \leq \epsilon$, any N-N bimodule H, and any unit vector $\xi \in H$, if $\|y\xi - \xi y\| < \delta$ for all $y \in F$, then there is a central vector $\eta \in H$ such that $\|\eta - \xi\| < K\delta$.

We call the finite set F a **Kazhdan set** for N. We call a presentation $N^\dagger$ of a property (T) factor N a **Kazhdan presentation** if there is a Kazhdan set for N amongst the rational points of $N^\dagger$.

Corollary

Suppose that N is a property (T) factor and $N^\dagger$ is a Kazhdan presentation of N. Then for any II_1 factor containing M and any presentation $M^\#$ of M, there is a computable spectral gap function for $(M^\#, N^\dagger)$.

Kazhdan presentations

Fact (Connes and Jones)

Suppose that N is a II_1 factor with property (T). Then there is $\epsilon > 0$, finite $F \subseteq N$, and $K > 0$ such that, for any $\delta \leq \epsilon$, any N-N bimodule H, and any unit vector $\xi \in H$, if $\|y\xi - \xi y\| < \delta$ for all $y \in F$, then there is a central vector $\eta \in H$ such that $\|\eta - \xi\| < K\delta$.

We call the finite set F a **Kazhdan set** for N. We call a presentation $N^\dagger$ of a property (T) factor N a **Kazhdan presentation** if there is a Kazhdan set for N amongst the rational points of $N^\dagger$.

Corollary

Suppose that N is a property (T) factor and $N^\dagger$ is a Kazhdan presentation of N. Then for any II_1 factor containing M and any presentation $M^\#$ of M, there is a computable spectral gap function for $(M^\#, N^\dagger)$.

Kazhdan presentations

Fact (Connes and Jones)

Suppose that N is a II_1 factor with property (T). Then there is $\epsilon > 0$, finite $F \subseteq N$, and $K > 0$ such that, for any $\delta \le \epsilon$, any N-N bimodule H, and any unit vector $\xi \in H$, if $\|y\xi - \xi y\| < \delta$ for all $y \in F$, then there is a central vector $\eta \in H$ such that $\|\eta - \xi\| < K\delta$.

We call the finite set F a **Kazhdan set** for N. We call a presentation $N^\dagger$ of a property (T) factor N a **Kazhdan presentation** if there is a Kazhdan set for N amongst the rational points of $N^\dagger$.

Corollary

Suppose that N is a property (T) factor and $N^\dagger$ is a Kazhdan presentation of N. Then for any II_1 factor containing M and any presentation $M^\#$ of M, there is a computable spectral gap function for $(M^\#, N^\dagger)$.

Macintyre's theorem reincarnated

First, given $m \in \mathbb{N}$, a II_1 factor M, and $u, a_1, \ldots, a_n, b \in M$, we set

$$\psi^M_{r,m}(u, \vec{a}, b) := \max\left(\|uu^* - 1\|_2, \max_{1 \le i \le m} \|[u, a_i]\|_2, 2r \dot{-} \|[u, y]\|_2\right).$$

We also set

$$\varphi^M_{r,m}(\vec{a}, b) := \inf_{u \in M_1} \psi^M_{r,m}(u, \vec{a}, b).$$

Lemma

Suppose that $M^\#$ is $\mathbf{D}$-computable. Then for each rational number r and each $m \in \mathbb{N}$, $\psi^M_{r,m} : (M^\#)^{m+2} \to \mathbb{C}$ is $\mathbf{D}$-computable.

Macintyre's theorem reincarnated

First, given $m \in \mathbb{N}$, a II_1 factor M, and $u, a_1, \ldots, a_n, b \in M$, we set

$$\psi^M_{r,m}(u, \vec{a}, b) := \max\left(\|uu^* - 1\|_2, \max_{1 \le i \le m} \|[u, a_i]\|_2, 2r \dot{-} \|[u, y]\|_2 \right).$$

We also set

$$\varphi^M_{r,m}(\vec{a}, b) := \inf_{u \in M_1} \psi^M_{r,m}(u, \vec{a}, b).$$

Lemma

*Suppose that $M^\#$ is **D**-computable. Then for each rational number r and each $m \in \mathbb{N}$, $\psi^M_{r,m} : (M^\#)^{m+2} \to \mathbb{C}$ is **D**-computable.*

Macintyre's theorem reincarnated (continued)

The following lemma involves a little "functional calculus" yoga.

Lemma

Suppose that $f : \mathbb{N} \to \mathbb{N}$ is a spectral gap function for $(M^{\#}, N^{\dagger})$. Then there is a function $f' : \mathbb{N} \to \mathbb{N}$, computable from any oracle computing f, so that, for any n and $r > 0$, if $\varphi^M_{r,f'(n)}(x_1, \ldots, x_{f'(n)}, b) < 2^{-f'(n)}$, then $d(b, N) \geq r - 2^{-n}$.

The following is proven in a manner similar to Macintyre's original argument.

Lemma

Suppose that M is e.c., that N is a subfactor of M, and that $(N, (x_n)_{n\in\mathbb{N}})$ is a presentation of N. If $b \in M$ is such that $d(b, N) = r$, then $\varphi^M_{r,m}(\vec{x}, b) = 0$ for all $m \in \mathbb{N}$.

Macintyre's theorem reincarnated (continued)

The following lemma involves a little "functional calculus" yoga.

Lemma

Suppose that $f : \mathbb{N} \to \mathbb{N}$ *is a spectral gap function for* $(M^{\#}, N^{\dagger})$*. Then there is a function* $f' : \mathbb{N} \to \mathbb{N}$*, computable from any oracle computing* f*, so that, for any* n *and* $r > 0$*, if* $\varphi^M_{r,f'(n)}(x_1, \ldots, x_{f'(n)}, b) < 2^{-f'(n)}$*, then* $d(b, N) \geq r - 2^{-n}$*.*

The following is proven in a manner similar to Macintyre's original argument.

Lemma

Suppose that M *is e.c., that* N *is a subfactor of* M*, and that* $(N, (x_n)_{n\in\mathbb{N}})$ *is a presentation of* N*. If* $b \in M$ *is such that* $d(b, N) = r$*, then* $\varphi^M_{r,m}(\vec{x}, b) = 0$ *for all* $m \in \mathbb{N}$*.*

Main theorem

Theorem

Suppose that N is a w-spectral gap subfactor of the e.c. factor M and that M and N have presentations $M^\#$ and $N^\dagger$ respectively so that the pair $(M^\#, N^\dagger)$ is a **D***-computable pair for some oracle* **D***. Further suppose that there is a* **D***-computable spectral gap function for $(M^\#, N^\dagger)$. Then $E_N : M^\# \to N^\dagger$ is* **D***-computable.*

Corollary

Suppose that M is e.c. and N is a property (T) subfactor of M. Let $N^\dagger$ be a Kazhdan presentation of N. If $(M^\#, N^\dagger)$ is a **D***-computable pair, then $E_N : M^\# \to N^\dagger$ is* **D***-computable.*

Main theorem

Theorem

Suppose that N is a w-spectral gap subfactor of the e.c. factor M and that M and N have presentations $M^{\#}$ and $N^{\dagger}$ respectively so that the pair $(M^{\#}, N^{\dagger})$ is a **D***-computable pair for some oracle* **D***. Further suppose that there is a* **D***-computable spectral gap function for $(M^{\#}, N^{\dagger})$. Then $E_N : M^{\#} \to N^{\dagger}$ is* **D***-computable.*

Corollary

Suppose that M is e.c. and N is a property (T) subfactor of M. Let $N^{\dagger}$ be a Kazhdan presentation of N. If $(M^{\#}, N^{\dagger})$ is a **D***-computable pair, then $E_N : M^{\#} \to N^{\dagger}$ is* **D***-computable.*

Proof of the main theorem

- Suppose we are given a rational point p of $M^\#$ and $k \in \mathbb{N}$. We use **D** to compute $d(p, N^\#)$ to within 2^{-k} using two machines.
- On the first machine, we start approximately computing $d(i(p'), p)$, where p' ranges over rational points of $N^\dagger$. This is done by finding rational $p'' \in M^\#$ such that $d(i(p'), p'')$ is small, and then computing $d(p'', p)$ approximately. This machine thus enumerates upper bounds for $d(p, N^\#)$.
- On the second machine, use **D** to compute $f'(k)$ and then start computing $\psi_{r,f'(k)}(u, \vec{p}, b)$ for rational points u and rational numbers r. If we see $\psi^M_{r,f'(k)}(u, \vec{p}, b) < 2^{-f'(k)}$, then we know that $d(b, N) \geq r - 2^{-k}$.
- We then wait until there is a rational number $r > 0$ so that the first machine tells us that $d(b, N) < r$ and the second machine tells us that $d(b, N) \geq r - 2^{-k}$. By the "Macintyre lemma", the latter is guaranteed to happen.

Proof of the main theorem

- Suppose we are given a rational point p of $M^{\#}$ and $k \in \mathbb{N}$. We use **D** to compute $d(p, N^{\#})$ to within 2^{-k} using two machines.
- On the first machine, we start approximately computing $d(i(p'), p)$, where p' ranges over rational points of $N^{\dagger}$. This is done by finding rational $p'' \in M^{\#}$ such that $d(i(p'), p'')$ is small, and then computing $d(p'', p)$ approximately. This machine thus enumerates upper bounds for $d(p, N^{\#})$.
- On the second machine, use **D** to compute $f'(k)$ and then start computing $\psi_{r,f'(k)}(u, \vec{p}, b)$ for rational points u and rational numbers r. If we see $\psi^{M}_{r,f'(k)}(u, \vec{p}, b) < 2^{-f'(k)}$, then we know that $d(b, N) \geq r - 2^{-k}$.
- We then wait until there is a rational number $r > 0$ so that the first machine tells us that $d(b, N) < r$ and the second machine tells us that $d(b, N) \geq r - 2^{-k}$. By the "Macintyre lemma", the latter is guaranteed to happen.

Proof of the main theorem

- Suppose we are given a rational point p of $M^\#$ and $k \in \mathbb{N}$. We use **D** to compute $d(p, N^\#)$ to within 2^{-k} using two machines.
- On the first machine, we start approximately computing $d(i(p'), p)$, where p' ranges over rational points of $N^\dagger$. This is done by finding rational $p'' \in M^\#$ such that $d(i(p'), p'')$ is small, and then computing $d(p'', p)$ approximately. This machine thus enumerates upper bounds for $d(p, N^\#)$.
- On the second machine, use **D** to compute $f'(k)$ and then start computing $\psi_{r,f'(k)}(u, \vec{p}, b)$ for rational points u and rational numbers r. If we see $\psi^M_{r,f'(k)}(u, \vec{p}, b) < 2^{-f'(k)}$, then we know that $d(b, N) \geq r - 2^{-k}$.
- We then wait until there is a rational number $r > 0$ so that the first machine tells us that $d(b, N) < r$ and the second machine tells us that $d(b, N) \geq r - 2^{-k}$. By the "Macintyre lemma", the latter is guaranteed to happen.

Proof of the main theorem

- Suppose we are given a rational point p of $M^{\#}$ and $k \in \mathbb{N}$. We use **D** to compute $d(p, N^{\#})$ to within 2^{-k} using two machines.
- On the first machine, we start approximately computing $d(i(p'), p)$, where p' ranges over rational points of $N^{\dagger}$. This is done by finding rational $p'' \in M^{\#}$ such that $d(i(p'), p'')$ is small, and then computing $d(p'', p)$ approximately. This machine thus enumerates upper bounds for $d(p, N^{\#})$.
- On the second machine, use **D** to compute $f'(k)$ and then start computing $\psi_{r,f'(k)}(u, \vec{p}, b)$ for rational points u and rational numbers r. If we see $\psi^{M}_{r,f'(k)}(u, \vec{p}, b) < 2^{-f'(k)}$, then we know that $d(b, N) \geq r - 2^{-k}$.
- We then wait until there is a rational number $r > 0$ so that the first machine tells us that $d(b, N) < r$ and the second machine tells us that $d(b, N) \geq r - 2^{-k}$. By the “Macintyre lemma”, the latter is guaranteed to happen.

References

- ISAAC GOLDBRING, *Oracle computability of conditional expectations onto subfactors*, New York Journal of Mathematics, Volume 27 (2021), 1085-1095.