Dynamical systems: moving from qualitative to computable results

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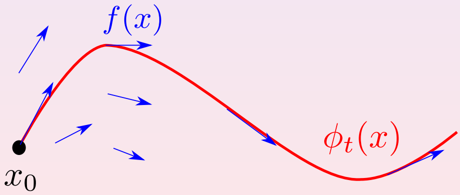
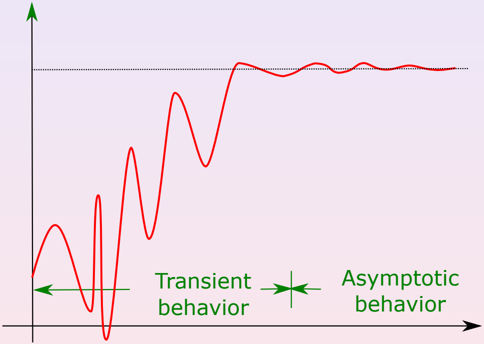
Some background

Given an ordinary differential equation (ODE) $x' = f(x)$, where $f : E \subseteq \mathbb{R}^2 \to \mathbb{R}^2$:

- f is the vector field
- $\phi(\cdot, x_0)$ is the solution curve of the ODE which starts with $x(0) = x_0$ ($\phi(t, x_0) = x(t)$)
- the planar flow is the set of all solution curves in E.

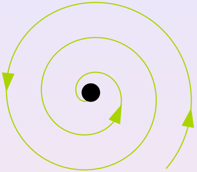


In dynamical systems theory it is usual to be interested in the asymptotic behavior of a system $x' = f(x)$.

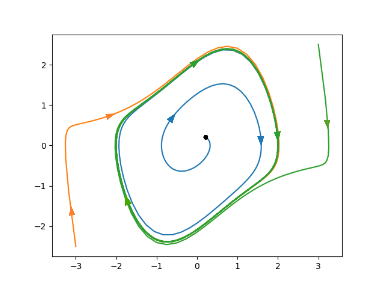


The limit (or non-wandering) set of $x' = f(x)$ is formed by the "steady states" of the system. Examples of constituents of the limit set are:

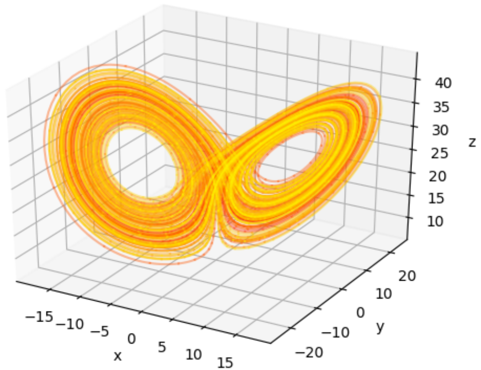
- Equilibrium points (solutions of $f(x) = 0$)



- Periodic orbits (limit cycles)



- Strange attractors



- **Problem:** In general it is very hard (if not impossible) to obtain formulas which describe precisely the asymptotic behavior of a given dynamical system.
- **Possible approaches:**
  - Qualitative approaches
  - Statistical approaches
  - Numerical methods

Can computable analysis help improve those techniques?

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Can computable analysis help improve those techniques?

Main problem of this talk

- Can we compute the number of periodic orbits of a polynomial planar ODE?
- Can we compute (i.e draw with arbitrary accuracy on a computer screen) all these periodic orbits?

Hilbert's problems

In 1900 David Hilbert published a very influential list of 23 problems for the 20th century.

Hilbert's 10th problem

Find an algorithm to determine whether a given polynomial Diophantine equation with integer coefficients has an integer solution.

- Proved to be impossible by Yuri Matiyasevich in 1970, building on earlier work from Martin Davis, Hilary Putnam and Julia Robinson.
- Several problems from Hilbert's original list remain open, including Hilbert's 16th problem.

Hilbert's 16th problem

Hilbert's 16th problem

- An investigation of the relative positions of the branches of real algebraic curves of degree n (and similarly for algebraic surfaces).
- The determination of the upper bound for the number of limit cycles in two-dimensional polynomial vector fields of degree n and an investigation of their relative positions

- Hilbert's 16th problem consists on two parts from different branches of mathematics.
- Both parts remain unsolved.
- In this talk we will consider the second part of Hilbert's 16th problem.

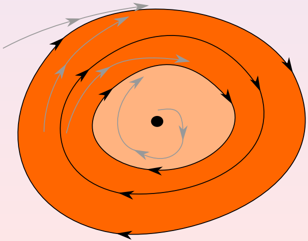
The two dimensional case

In two dimensions the situation is a bit simpler.

Theorem

Any polynomial vector field has at most a finite number of limit cycles in $\mathbb{R}^2$

- The Poincaré-Bendixson theorem rules out strange attractors on two-dimensional (planar) vector fields.
- Yet the two dimensional case is still hard - Hilbert's 16th Problem concerns two dimensional flows!



Hilbert's 16th problem – 2nd part

The determination of the upper bound for the number of limit cycles in two-dimensional polynomial vector fields of degree n and an investigation of their relative positions

- This problem remains open even for the simplest nonlinear case, where $n = 2$;
- It is known that there are planar quadratic polynomial vector fields with 4 periodic orbits.
- Upper bounds are hard!

Problem

Is the operator which maps a function $p : \mathbb{R}^2 \to \mathbb{R}^2$ with polynomial components of a given degree, to the number of limit cycles of $x' = p(x)$ computable?

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Theorem (The operator is non-computable...)

The operator which maps a function $p : \mathbb{R}^2 \to \mathbb{R}^2$ *with cubic polynomial components to the number of limit cycles of* $x' = p(x)$ *is noncomputable.*

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Theorem (... but it is computable over a dense subset)

Let $\mathbb{D} \subseteq \mathbb{R}^2$ be the closed unit disk and let SS_2 be the subset of $\mathcal{X}(\mathbb{D})$ consisting of all C^1 structurally stable vector fields $f : \mathbb{D} \to \mathbb{R}^2$. Then the operator which maps $f \in SS_2$ to the number of limit cycles of $x' = p(x)$ is (uniformly) computable. Meanwhile, the algorithm which produces the computation can depict the limit cycles with arbitrarily high precision.

Applications to Hilbert's 16th problem

Theorem (Finding sharp upper bounds is a non-computable problem)

There is a family of polynomial systems $x' = p(x)$ *which does not have a computable sharp upper bound on the number of its limit cycles.*

Theorem

$Hilbert_{16}(\mathcal{P} \cap SS_2) \leq HALT$.

Theorem

There are dense subsets A, B *of* $\mathcal{P}$, *where* B *is also open, such that* $B \subseteq A \subseteq \mathcal{P}$ *and* $Hilbert_{16}(A) \leq HALT$.

Representing real data

- Each real number $x \in \mathbb{R}$ can be represented by a function $f : \{0\}^* \to \mathbb{Q}$ with the property that $|f(n) - r| \leq 2^{-n}$.

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- A function $f : \{0\}^* \to \{0,1\}^*$ represents a (C^k-) function on a compact $K \subseteq \mathbb{R}^j$ if $f(n)$ encodes the rational coefficients of a polynomial P_n such that $d^k(f, P_n) \leq 2^{-n}$, where

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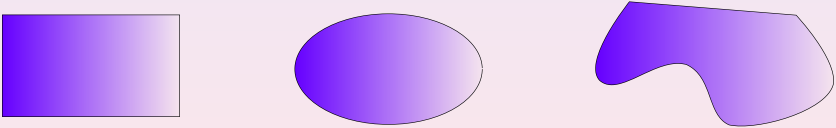
- A function $F : \mathbb{R} \to \mathbb{R}$ is computable if there is an oracle Turing machine M such that given a function ϕ representing a real number x, M^ϕ represents $f(x)$.

- Let $A \subseteq \mathbb{R}^n$ be a closed non-empty set. Then A is computable if the distance function defined by $d_A(x) = \inf_{y \in A} \|x - y\|$ is computable.

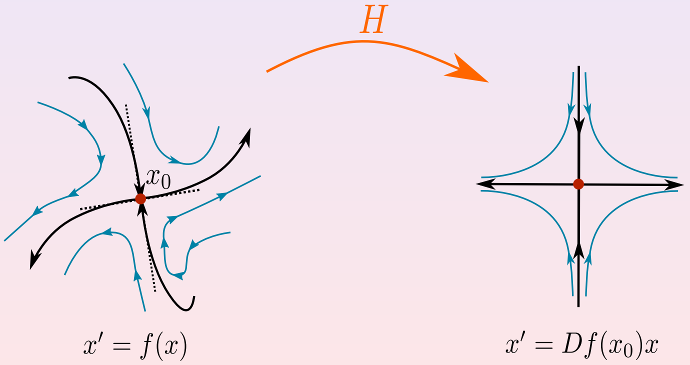
Theorem

Let $C \subset \mathbb{R}^n$ be a compact set. Then C is computable iff it can be drawn on a computer screen with arbitrary accuracy for $n = 1, 2$.



Example

Hartman-Grobman theorem: Near a hyperbolic equilibrium point x_0 of a nonlinear ODE $x' = f(x)$, there is a computable homeomorphism H such that $H \circ \phi = L \circ H$, where ϕ is the solution to the ODE and L is the solution to its linearization $x' = Df(x_0)\,x$.



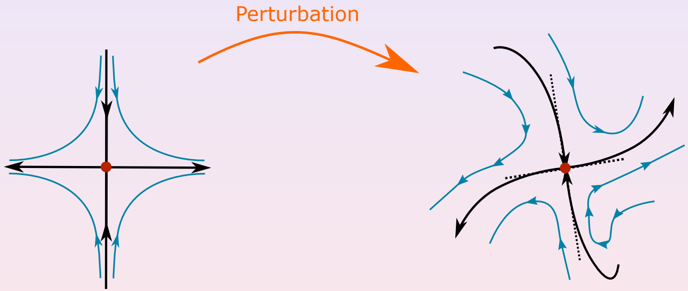
Question (Part of the 7th open problem of Pour-El and Richards' book): Is H and the neighborhood where the Hartman-Grobman theorem holds computable?

Theorem (Dumas, G., Zhong, 2012)

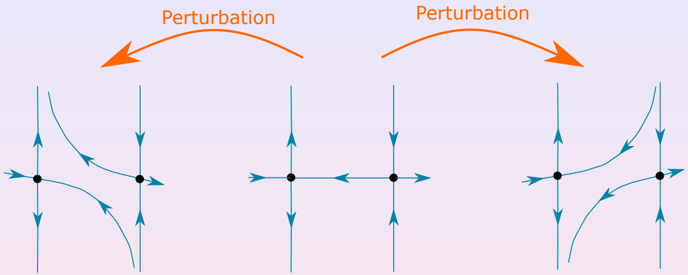
YES, the Hartman-Grobman theorem admits a computable version.

The positive result

- A celebrated result of Peixoto (following work from Pontryagin and Andronov) describes the typical asymptotic shape of the flow in two dimensions.



Structurally stable system



Structurally unstable system

Theorem (Peixoto (1962), Pontryagin and Andronov (1937))

Let f be a C^1 vector field defined on a compact two-dimensional differentiable manifold $K \subseteq \mathbb{R}^2$. Then f is structurally stable on K if and only if:

1. *The number of equilibria (i.e. zeros of f) and of periodic orbits is finite and each is hyperbolic;*
2. *There are no trajectories connecting saddle points, i.e. there are no saddle connections;*
3. *The non-wandering set $NW(f)$ consists only of equilibria and periodic orbits.*

Moreover, if K is orientable, the set of structurally stable vector fields in $C^1(K)$ is an open, dense subset of $C^1(K)$. Similar results hold for $\mathbb{D} = \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$, assuming that the vector fields always point inwards on the boundary of $\mathbb{D}$.

Why structurally stable systems?

- Structurally stable systems over $\mathbb{D}$ are dense;
- Structural stability implies robustness to perturbations;
- This implies that we can use approximations and still approximate the exact result.

Proof of the positive result

We first prove that the limit set (equilibrium points + periodic orbits) can be depicted with arbitrarily high accuracy.

Theorem

The operator $\Psi : SS_2 \to \mathcal{K}(\mathbb{D})$, $f \mapsto NW(f)$ *of* $x' = p(x)$, *is computable.*

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To prove this result we need to compute

- The set of zeros $Zero(f) = \{x \in \mathbb{D} : f(x) = 0\}$
- The set of periodic orbits $Per(f)$

Computing zeros of a function

Theorem (G., Zhong (2021))

Let $\mathcal{Z}(K) = \{f \in C^1(K) : f(\alpha) = 0 \Rightarrow \det(Df(\alpha) \neq 0 \text{ and } \alpha \notin \partial K)\}$.
Then the following are (uniformly) computable

- *The operator which maps* $f \in \mathcal{Z}(K)$ *to* $Zero(f)$
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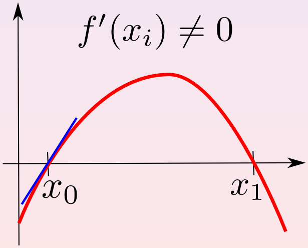


The idea to prove this theorem is the following.

- To compute $Zero(f)$ with accuracy $1/k$, start from $l = k$ and cover $\mathbb{D}$ with side-length $1/l$ square pixels.
- For each pixel s, compute $d(0, f(s))$ and $\min_{x\in s} |\det Df(x)|$, increase l if necessary until either $d(0, f(s)) > 2^{-l}$ or $\min_{x\in s} |\det Df(x)| > 2^{-l}$ after finitely many increments.
- If $d(0, f(s)) > 2^{-l}$, then there is no equilibrium inside s
- If $\min_{x\in s} |\det Df(x)| > 2^{-l}$, then use the elements from the proof of the inverse function theorem to either locate the squares contained in s such that each hosts a unique equilibrium or else increase l and repeat the process on smaller pixels contained in s.
- This also yields an algorithm to compute $\#Zero(f)$.

Computing periodic orbits

Theorem (G., Zhong)

The following operators are (uniformly) computable:

- $\Psi_{per} : SS_2 \to \mathcal{K}(\mathbb{D}),\ f \mapsto Per(f);$
- $\Psi_{\#per} : SS_2 \to \mathbb{N},\ f \mapsto \textit{number of periodic orbits of } x' = p(x).$

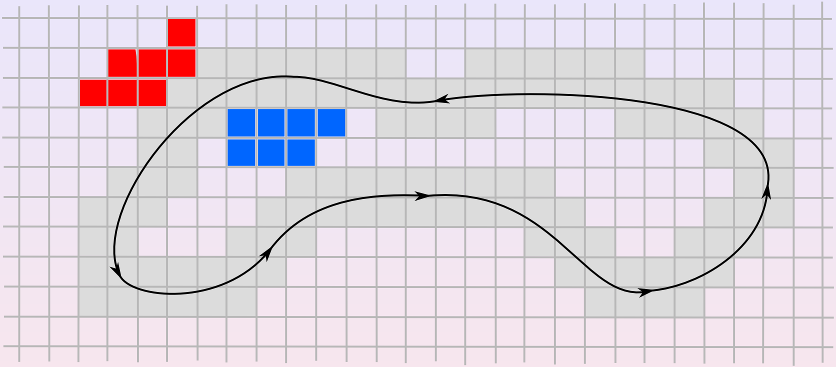
In other words, we can compute the exact number of periodic orbits for structurally stable systems and depict those periodic orbits with arbitrary accuracy.

Using e.g. Euler's method, we can devise a new computable function `TimeEvolution` that receives as input some compact set $D \subseteq \mathbb{D}$, f and some rational numbers $0 < \epsilon < 1$ and $T > 0$, with the following properties:

- $\phi_T(D) \subseteq \texttt{TimeEvolution}(D, \epsilon, T) \subseteq \phi_T(D) + \epsilon B(0,1)$. In particular this implies that $d_H(\phi_T(D), \texttt{TimeEvolution}(D, \epsilon, T)) \leq \epsilon$.
- The computation of `TimeEvolution` with input (D, ϵ, T) halts in finite time.

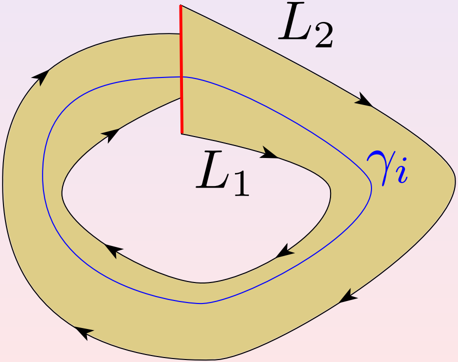
Then to compute $Per(f)$ we repeatedly iterate $\texttt{TimeEvolution}(\mathbb{D}, \epsilon, T)$ with increasing time T $(-T)$ and decreasing ϵ.

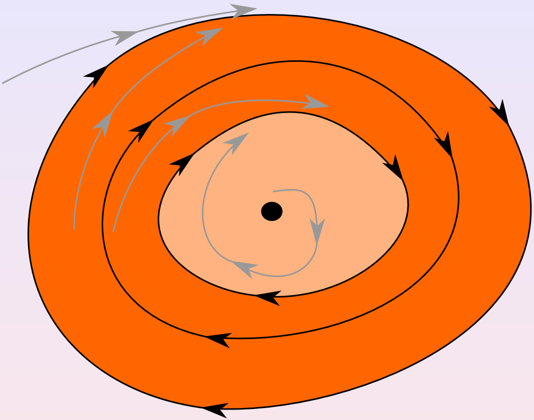
- If there are no saddle points, the method will converge to $A = Per(f) \cup Zero(f)$
- We then check each connected component of $A - Zero(f)$ to see if it has a "donut" shape, using a coloring algorithm
- If the previous test succeeds, we compute the maximum cross-section of the donut – this will be the accuracy of the approximation to the periodic orbit



The coloring algorithm

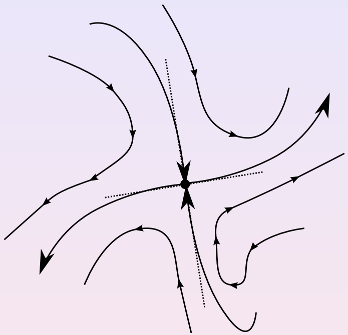
- We can compute a Poincaré section and a Poincaré map on each connected component of $A - Zero(f)$. We can also be sure that no periodic orbit lies inside $Zero(f)$ using an effective version of the Hartman-Grobman theorem
- By counting the number of fixed points of each Poincaré map, we are able to count the number of periodic orbits





Nested orbits

The case of saddle points



- The flow may take an arbitrary amount of time to pass near a zero of f.
- This does not allow for the use of uniform convergence in time to $Per(f) \cup Zero(f)$.

Solution: linearize the flow near an equilibrium point using an effective version of the Hartman-Grobman theorem:

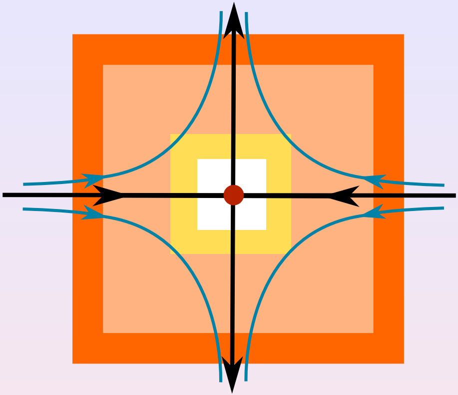
Theorem (G. , Zhong and Dumas, 2012)

There is a computable map $\Theta : \mathcal{F} \to \mathcal{O} \times \mathcal{O} \times C(\mathbb{R}^n; \mathbb{R}) \times C(\mathbb{R}^n; \mathbb{R}^n)$ *such that for any* $f \in \mathcal{F}$, $f \mapsto (U, V, \mu, H)$, *where*

(a) $H : U \to V$ *is a homeomorphism ;*

(b) *the unique solution* $x(t, \tilde{x}) = x(\tilde{x})(t)$ *to the initial value problem* $x' = f(x)$ *and* $x(0) = \tilde{x}$ *is defined on* $(-\mu(\tilde{x}), \mu(\tilde{x})) \times U$*; moreover,* $x(t, \tilde{x}) \in U$ *for all* $\tilde{x} \in U$ *and* $-\mu(\tilde{x}) < t < \mu(\tilde{x})$ *;*

(c) $H(x(t, \tilde{x})) = e^{Df(0)t} H(\tilde{x})$ *for all* $\tilde{x} \in U$ *and* $-\mu(\tilde{x}) < t < \mu(\tilde{x})$ *.*



- We can explicitly solve a linear ordinary differential equation and thus determine where the flow will leave some box defining a neighborhood of a saddle point where the effective Hartman-Grobman theorem is valid.
- Use this as the result of the flow which enters the box, and do not account time for this step – the overall algorithm will work

Proving non-computability of the operator

Theorem (The operator is non-computable...)

The operator which maps a function $p : \mathbb{R}^2 \to \mathbb{R}^2$ *with cubic polynomial components to the number of periodic orbits of* $x' = p(x)$ *is noncomputable.*

- In short the proof of this theorem is based on the fact that the operator is not continuous.

Lemma

The function

$$h(k) = \begin{cases} 1 & \text{if } TM_k \text{ halts on } k \\ 0 & \text{if } TM_k \text{ doesn't halt on } k \end{cases}$$

is not computable.

- By absurd, assume that the operator Θ which maps p to the number of limit cycles of $x' = p(x)$ is computable
- Let $g : \mathbb{N}^2 \to \mathbb{N}$ be the computable function defined as follows:

$$g(k,i) = \begin{cases} 1 & \text{if } TM_k \text{ halts in } \leq i \text{ steps on input } k \\ 0 & \text{otherwise} \end{cases}$$

- Let $G : \mathbb{N} \to [0,1]$, $G(k) = \sum_{i=1}^{\infty} g(k,i)2^{-i}$. Then G is also a computable function with $\sum_{i=1}^{n} g(k,i)$ being a rational approximation of $G(k)$ with accuracy 2^{-n}. Moreover, $0 < G(k) \leq 1$ if TM_k halts on k and $G(k) = 0$ if TM_k doesn't halt on k
- Now define a family of polynomial systems with parameters $G(k)$: $x' = p_k(x)$, where

$$p_{k,1}(x_1, x_2) = -x_2 + x_1(x_1^2 + x_2^2 - G(k))$$

and

$$p_{k,2}(x_1, x_2) = x_1 + x_2(x_1^2 + x_2^2 - G(k)).$$

- Since $G : \mathbb{N} \to [0, 1]$ is computable, so is the function $P : \mathbb{N} \to \mathcal{P}$, $k \mapsto p_k$. By assumption that Θ is computable, it follows that the composition $\Theta \circ P : \mathbb{N} \to \mathbb{N}$ is a computable function.
- In the polar coordinates, the system is converted to the following form: let $\theta = t$,

$$dr/dt = r(r^2 - G(k)), d(\theta)/dt = 1$$

  Thus, if TM_k doesn't halt on k, then $dr/dt = r^3$, and there is only one equilibrium point at the origin and no periodic orbit; if TM_k does halt on k, then $dr/dt = r(r^2 - G(k))$ and there is one periodic orbit and one equilibrium point.
- In other words,

$$\Theta \circ P(k) = \begin{cases} 1 & \text{if } TM_k \text{ halts on } k \\ 0 & \text{if } TM_k \text{ doesn't halt on } k. \end{cases}$$

- We arrive at a contradiction because it follows from the previous Lemma that $\Theta \circ P$ cannot be a computable function.

Theorem (Finding sharp upper bounds is a non-computable problem)

There is a family of polynomial systems $x' = p(x)$ *which does not have a computable sharp upper bound on the number of its limit cycles.*

Corollary (there is more than non-continuity to non-computability)

There exists a family of polynomials $\{p_n\}_{n\in\mathbb{N}}$ *and a value* $\delta > 0$ *such that* $\|p_n - p_m\| \geq \delta$ *whenever* $n \neq m$, *with the property that the operator which maps polynomials to the number of periodic orbits is still noncomputable over the set* $\{p_n : n \in \mathbb{N}\}$.

Theorem (Finding sharp upper bounds is a non-computable problem)

There is a family of cubic polynomial systems $x' = p(x)$ which does not have a computable sharp upper bound on the number of its limit cycles.

- The non-continuity idea no longer applies
- But techniques based on the previous non-computability result can be used to show this theorem.
- To prove this, let $\{p_k\}_{k\in\mathbb{N}}$ be the family of polynomial systems with parameters $G(k)$: $dx/dt = p_k(x,y)$, where $p_k(x,y) = (p_{k,1}(x,y), p_{k,2}(x,y))$,

$$p_{k,1}(x,y) = -y + x\Pi_{j=1}^{k}(x^2+y^2-\Sigma_{i=1}^{j} iG(i))$$
$$= -y + x(x^2+y^2-G(1))\cdots(x^2+y^2-(G(1)+2G(2)+\cdots$$

and

$$p_{k,2}(x,y) = x + y\Pi_{j=1}^{k}(x^2+y^2-\Sigma_{i=1}^{j} iG(i))$$

- In the polar coordinates, the system is converted to the following one: let $\theta = t$,

$$dr/dt = r\Pi_{j=1}^{k}(r^2 - \Sigma_{i=1}^{j} iG(i)), \text{ and } d(\theta)/dt = 1.$$

- By absurd, assume that there was a computable sharp upper bound f for the number of periodic orbits for this family of polynomial systems.
- It follows from the definition that the components of p_k have degree $1 + 2k$.
- $f(3)$ would yield a sharp upper bound for the number of periodic orbits of $x' = p(x)$ when $p = p_1$. In particular, if $f(3) = 0$, then $G(1) = 0$, which implies that TM_1 does not halt on input 1 and thus $h(1) = 0$; if $f(3) = 1$, then $G(1) > 0$ and thus $h(1) = 1$.
- For $k > 1$. We observe that if $h(k) = 0$, then $G(k) = 0$ and thus $f(1 + 2(k-1)) = f(1 + 2k)$; on the other hand, if $h(k) = 1$, then $G(k) > 0$ and $f(1 + 2k) = f(1 + 2(k-1)) + 1$.

- This observation together with the assumption that f is computable generates the following algorithm for computing $h(k)$ of (recall that h is a non-computable function)

$$h(k) = \begin{cases} 1 & \text{if } f(1+2k) = f(1+2(k-1)) + 1 \\ 0 & \text{if } f(1+2k) = f(1+2(k-1)) \end{cases}$$

for $k > 1$. We arrive at a contradiction.

More details can be found at:

D. Graça and N. Zhong, "Computing the exact number of periodic orbits for planar flows", to appear in *Transactions of the American Mathematical Society*

http://arxiv.org/abs/2101.07701

Thank you!

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