A construction of a λ- Poisson generic sequence

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Plan

Poisson Generic Sequences

Fix Ω an alphabet with b symbols.

Notation

Given $x \in \Omega^{\mathbb{N}}$, $\lambda \in \mathbb{R}_{>0}$, $i \in \mathbb{N}_0$ and $k \in \mathbb{N}$ we write $Z_{i,k}^{\lambda}(x)$ for the proportion of words of length k that occur exactly i times in the prefix of x of length $\lfloor \lambda b^k \rfloor$, that is:

$$Z_{i,k}^{\lambda}(x) = \frac{\#\{\omega \in \Omega^k : \omega \text{ occurs i times in } x[1 ... \lfloor \lambda b^k \rfloor]\}}{b^k}$$

Definition (Zeev Rudnick)

We say $x \in \Omega^{\mathbb{N}}$ is λ-**Poisson generic** if for every $i \in \mathbb{N}_0$:

$$\lim_{k \to \infty} Z_{i,k}^{\lambda}(x) = \frac{e^{-\lambda} \lambda^i}{i!}$$

x is **Poisson generic** if it is λ-Poisson generic for every $\lambda > 0$.

The problem

Question (Weiss)

Is it possible to give an explicit construction of a 1-Poisson generic sequence?

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Question

Can we give a “ more explicit” example?

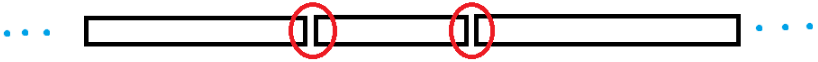
Main theorem

Theorem (Becher, S.H.) λ-Poisson explicit number

Let λ be a positive real number and Ω a b-symbol alphabet , $b \geq 3$. Let $(p_i)_{i \in \mathbb{N}_0}$ be a sequence of non-negative real numbers such that $\sum_{i \geq 0} p_i = 1$ and $\sum_{i \geq 0} i p_i = \lambda$. Then there is a construction of an infinite sequence x over alphabet Ω, which satisfies for every $i \in \mathbb{N}_0$,

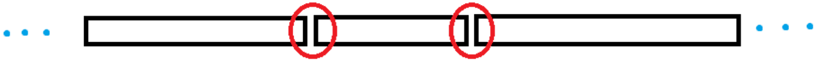
$$\lim_{k \to \infty} Z_{i,k}^{\lambda}(x) = p_i.$$

Concatenating big blocks



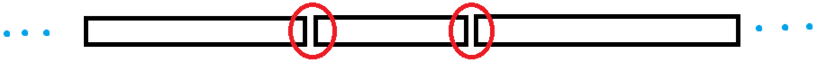
The construction: Core ideas

Concatenating big blocks



Where do we take the blocks from?

Concatenating big blocks



Where do we take the blocks from?

An infinite de Bruijn word

The construction

Definition

A **cyclic de Bruijn sequence of order** k is a sequence w of length b^k where each word of length k occurs exactly once in the circular sequence determined by w.

Theorem (Becher, Heiber)

For $b \geq 3$ there exists an infinite word x such that every prefix $x[1 \ldots b^k]$ is a cyclic de Bruijn word in base b of order k.

Definition

Given a sequence w of length b^k we say δ is a **block** in w if it is a subsequence of w and $|\delta| = b^j \leq b^k$ for some $j \in \mathbb{N}_0$.
We say that a block δ in w has **absolute length** $|\delta|$ and **relative length** $|\delta|b^{-k}$ **with respect to** w.

The construction

Fix A an infinite de Bruijn word in base b.

$$A = 012110022101020001112021222...$$

In step $k+1$ of the construction we will take blocks from $A[b^k + 1 \dots b^{k+1}]$:

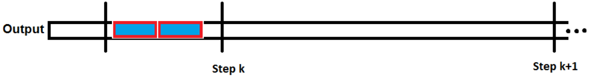
$$A = 012 \Big| 110022 \Big| 101020001112021222 \Big| ...$$

We always pick **non- overlapping** blocks.

Why infinite de Bruijn words?

$A = 012 \mid 110022 \mid 101020001\ 112\ 021222 \mid \ldots$

$Output = 011000222\ 101020001\ 101020001\ 112\ 112\ 112$



- If a block contributes a to $Z^{\lambda}_{2,k}$, then it contributes approximately with $\frac{1}{b}a$ to $Z^{\lambda}_{2,k+1}$.
- We can add the contributions to $Z^{\lambda}_{i,k}$ of distinct non-overlapping blocks that are repeated exactly i times.

The construction

Notation

Given a real number $y \in [0, 1)$, we write $\{y\}_k$ for the truncation to k digits of the unique base-b representation of y which does not end in an infinite tail of $(b-1)$'s.

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The construction. Step 1

The construction works by steps. Let x_k be the output of the construction after Step k. Start with x_0 equal to the empty word.
Step 1 For $i \geq 1$

$$\{p_i\}_1 = 0, c_i \text{ (base b expansion)}$$

For each c_i, $i \geq 1$, choose c_i blocks of relative length b^{-1} with respect to $A[1...b]$.
Concatenate the chosen blocks, in any order, where for every $i \geq 1$ each of the c_i selected blocks is repeated exactly i times.

The construction

An example: Take $p_0 = 0$, $p_1 = 1/2$, $p_2 = 5/18$, $p_3 = 2/9$, and $p_i = 0$ for $i \geq 4$. In this case $\lambda = 31/18$. Now fix $b = 3$, $\Omega = \{0, 1, 2\}$ and

$$A = 012110022101020001112021222...$$

$$p_1 = 0,1111...$$
$$p_2 = 0,0211...$$
$$p_3 = 0,0200...$$

Step 1:

$$A = \boxed{0}12 \Big| 110022 \Big| 1010200011120212222 \Big| ...$$

$$x_1 = \boxed{0}$$

The construction

Step k+1

Step k+1 For $i \geq 1$:

$$\left\{ \frac{b-1}{b} p_i \right\}_{g(k+1)} = 0, a_{i,1} a_{i,2} .. a_{i,g(k+1)} \text{ (base b expansion)}$$

where $a_{i,j} \in \{0, 1, 2, ..., b-1\}$.

- We select blocks in $A_{k+1}[b^k + 1 ... b^{k+1}]$ in the following manner:
  For each $a_{i,j}$, we choose $a_{i,j}$ blocks of relative length b^{-j} with respect to $A[1 ... b^{k+1}]$. All the selected blocks should be non-overlapping.
- The construction appends these blocks to x_k. For every $i \geq 1$, $j \leq g(k+1)$, each of the $a_{i,j}$ selected blocks is repeated exactly i times.

The construction

Example: We set $g(k) = k$

$$\frac{2}{3}p_1 = 0,1000\ldots$$
$$\frac{2}{3}p_2 = 0,0120\ldots$$
$$\frac{2}{3}p_3 = 0,0110\ldots$$

Step 3:

$A = 012 \mid 110022 \mid$ 101020001 | 112 | 021 | 2 | 2 | 2 $\mid \ldots$

$x_3 =$ 011000222 101020001 | 112 | 112 | 021 | 021 | 021 | 2 | 2 | 2 | 2 | 2 | 2 | 2

The construction

Definition

We refer to each of the chosen blocks of A as **constituent segments in the output** x_{k+1}.
We say that the concatenation of i-many copies of a constituent segment corresponding to $a_{i,j}$ is a **run segment** in the output.

$x_3 = 011000222$ | 101020001 | 112 | 112 | 021 | 021 | 021 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |

In this case, | 112 | 112 | is the run segment corresponding to the constituent segment | 112 |, and | 021 | 021 | 021 | is the run segment corresponding to the constituent segment | 021 |.

Correctness

Theorem

Let x be the infinite word output by the algorithm. Then $\lim_{k\to\infty} Z^{\lambda}_{i,k}(x) = p_i$ for every $i \geq 0$.

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Definition

For every $i \geq 1$, p_i^k denotes the sum of the relative lengths with respect to $A[1...b^k]$ of all constituent segments in the output x_k that are repeated exactly i times.
We define $p_0^k = 1 - \sum_{i\geq 1} p_i^k$.

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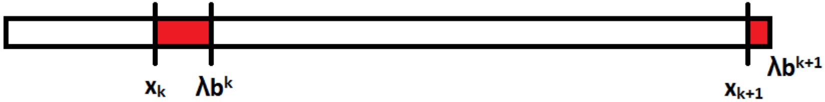
Proof: We will use p_i^k as an approximation for $Z^{\lambda}_{i,k}(x)$.
We are assuming the contribution of a constituent segment to the numerator of $Z^{\lambda}_{i,k}(x)$ is its absolute length.

Correctness: Bounding the error

First error



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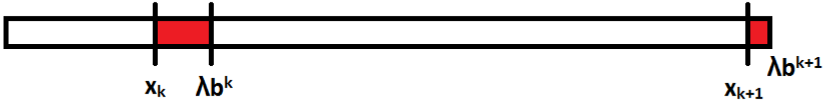

Lemma

Let x_k be the word output by the construction after step k. Then,

$$\lim_{k\to\infty} \frac{\left|\lfloor \lambda b^k \rfloor - |x_k|\right|}{b^k} = 0.$$

Second error

Let B_k be the number of run segments in the output x_k.
Error $\sim kB_k$

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Error $\sim kB_k$

Lemma

If we set $g(k) = \lceil k/2 \rceil$ then the quantities B_k satisfy

$$\lim_{k \to \infty} \frac{kB_k}{b^k} = 0.$$

Correctness

To conclude:

Lemma

For every $i \in \mathbb{N}_0$, $\lim_{k\to\infty} p_i^k = p_i$. *In fact, for every* $i \geq 1$, $k \geq 1$, *the following estimation holds,*

$$p_i - \frac{k}{b^{g(k)}} \leq p_i^k \leq p_i.$$

Limitations

The construction does not allow us to generate a Poisson generic sequence.

Suppose we construct x for $\lambda = 1$. Then the frequencies for $\lambda = 1/b$, $i \geq 1$, satisfy,

$$\lim_{k \to \infty} Z^{1/b}_{i,k+1}(x) - \frac{1}{b} Z^{1}_{i,k}(x) = 0.$$

But this relation does not hold in the case of the probability mass function of the Poisson distribution:

$$e^{-1/b} \frac{1}{b^i i!} \neq e^{-1} \frac{1}{b i!}.$$

Further results

Theorem (Becher, S.H.)

Let Ω be a b-symbol alphabet, $b \geq 2$, and let $x \in \Omega^{\mathbb{N}}$. We fix a positive real number λ and define for every $i \in \mathbb{N}_0$ the numbers $p_i = \liminf_{k\to\infty} Z^{\lambda}_{i,k}(x)$. If the numbers p_i satisfy $\sum_{i\geq 0} ip_i = \lambda$ then x is normal to base b.

Corollary

The presented construction yields infinitely many Borel normal sequences which are not λ-Poisson generic.

Thanks!

Thank you ! Questions?