Further properties of nearly computable numbers

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Convention: A "sequence" in this talk is always a sequence of rational numbers.

Definition

1. A sequence $(x_n)_n$ *converges computably* to a real number x if there is a computable function $u : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$ and for all $i \geq u(n)$ the inequality $|x - x_i| < 2^{-n}$ holds.

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2. A real number is called *computable* if there is a computable sequence converging computably to it.

Definition

1. A sequence $(x_n)_n$ is called *nearly computably Cauchy* if for every computable increasing function $s : \mathbb{N} \to \mathbb{N}$ the sequence $(x_{s(n+1)} - x_{s(n)})_n$ converges computably to zero.

Problem

Is every nearly computably Cauchy sequence a Cauchy sequence?

Lemma

Let $(x_n)_n$ *be a computable sequence such that for every computable increasing function* $s : \mathbb{N} \to \mathbb{N}$ *the sequence* $(x_{s(n+1)} - x_{s(n)})_n$ *converges to zero. Then* $(x_n)_n$ *converges.*

Lemma

Let $(x_n)_n$ be a computable sequence such that for every computable increasing function $s : \mathbb{N} \to \mathbb{N}$ the sequence $(x_{s(n+1)} - x_{s(n)})_n$ converges to zero. Then $(x_n)_n$ converges.

Proof.

Suppose that $(x_n)_n$ does not converge. Then it is also not a Cauchy sequence. So there exists a rational number $\varepsilon > 0$ and an infinite index chain $i_0 < j_0 < i_1 < j_1 < i_2 < j_2 < \dots$ with $|x_{i_n} - x_{j_n}| \geq \varepsilon$ for all $n \in \mathbb{N}$. Since $(x_n)_n$ is computable, we can effectively look for those indices. So there exists a computable increasing function $s : \mathbb{N} \to \mathbb{N}$ with $s(2n) = i_n$ and $s(2n+1) = j_n$ for all $n \in \mathbb{N}$. But then, the sequence $\left(x_{s(n+1)} - x_{s(n)}\right)_n$ does not converge to zero, which is a contradiction. So $(x_n)_n$ converges indeed. □

Theorem

There exists an increasing unbounded sequence which is nearly computably Cauchy.

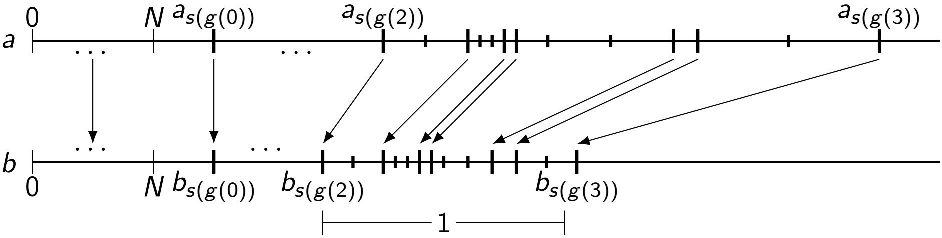

Figure: An illustration of the construction in the proof

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2. A real number is called *nearly computable* if there is a computable sequence converging nearly computably to it.

Every computable number is nearly computable, but the converse is not true.

Theorem

Let be $k \geq 1$ *a natural number,* $U \subseteq \mathbb{R}^k$ *an open subset,* $f : U \to \mathbb{R}$ *a computable function and* $x_1, \ldots, x_k$ *nearly computable numbers with* $(x_1, \ldots, x_k) \in U$.
Then $f(x_1, \ldots, x_k)$ *is also nearly computable.*

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Then $f(x_1, \ldots, x_k)$ is also nearly computable.

Thus, the set of the nearly computable numbers is closed under total computable functions.

Definition

For a natural number $b \geq 2$ and a set $A \subseteq \mathbb{N}$ we define:

$$b^{-A} := \sum_{n \in A} b^{-n}$$

Proposition

For a set $A \subseteq \mathbb{N}$ the following are equivalent:

1. *A is computable.*
2. *4^{-A} is computable.*
3. *4^{-A} is nearly computable.*

Theorem

Let $\nu_{\mathbb{Q}}$ be a standard enumeration of the rational numbers. Let us define the function $F : \mathbb{R} \to \mathbb{R}$ with

$$F(x) := \sum_{\substack{k \in \mathbb{N}, \\ \nu_{\mathbb{Q}}(k) < x}} 4^{-k}$$

for all $x \in \mathbb{R}$, and let $f : \mathbb{R} \setminus \mathbb{Q} \to \mathbb{R}$ be the restriction of F to the irrational numbers. Then we have:

1. *f is computable.*
2. *If a number x is not computable, then $f(x)$ is not nearly computable.*

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1. *f is computable.*
2. *If a number x is not computable, then $f(x)$ is not nearly computable.*

Thus, the set of the nearly computable numbers is not closed under partial computable functions.

Definition

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Theorem (Downey, LaForte (2002), Stephan, Wu (2005))

There exists a left-computable number which is nearly computable but not computable.

Proposition

For a set $A \subseteq \mathbb{N}$ *we have:*

1. A *is computable if and only if* 2^{-A} *is computable.*
2. *If* A *is computably enumerable, then* 2^{-A} *is left-computable.*

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It is well-known that the converse of the second statement is not true.

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It is well-known that the converse of the second statement is not true.

Definition

A real number $x \in [0; 2]$ is called *strongly left-computable* if there exists a computably enumerable set $A \subseteq \mathbb{N}$ with $x = 2^{-A}$.

Proposition

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A real number $x \in [0; 2]$ is called *strongly left-computable* if there exists a computably enumerable set $A \subseteq \mathbb{N}$ with $x = 2^{-A}$.

Proposition (Downey, Hirschfeldt, LaForte 2004)

Let x be a strongly left-computable number that is nearly computable. Then x is even computable.

Definition

A set $A \subseteq \mathbb{N}$ is called *hyperimmune* if it is infinite and there is no computable function $f : \mathbb{N} \to \mathbb{N}$ with $A \cap \{n, \ldots, f(n)\} \neq \emptyset$ for all $n \in \mathbb{N}$.

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Definition (Stephan, Wu 2005)

A set $A \subseteq \mathbb{N}$ is called *strongly Kurtz random* if there is no computable function $f : \mathbb{N} \to \mathbb{N}$ with $K(A \upharpoonright f(n)) < f(n) - n$ for all $n \in \mathbb{N}$.

Theorem (Stephan, Wu 2005)

Let $A \subseteq \mathbb{N}$ be a set such that 2^{-A} is left-computable and nearly computable, but not computable. Then we have:

1. *A is hyperimmune (and hence, not Martin-Löf random).*
2. *A is strongly Kurtz random (and hence, not K-trivial).*

These theorems can be strengthened: the assumption that the number is left-computable can be omitted.

Theorem

Let $A \subseteq \mathbb{N}$ be a set such that 2^{-A} is nearly computable, but not computable. Then we have:

1. *A is hyperimmune (and hence, not Martin-Löf random).*
2. *A is strongly Kurtz random (and hence, not K-trivial).*

Definition (Maass 1983)

A set $A \subseteq \mathbb{N}$ is called *promptly simple* if it satisfies the following properties:

- A is computably enumerable.
- The complement $\overline{A}$ is infinite.
- There exists a computable function $h : \mathbb{N} \to \mathbb{N}$ such that, for every $e \in \mathbb{N}$, if W_e is infinite then there exist $s \in \mathbb{N}$ and $x \in \mathbb{N}$ with $x \in (W_e[s+1] \setminus W_e[s]) \cap A[h(s)]$.

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Theorem (Downey, LaForte 2002)

Let $A, B \subseteq \mathbb{N}$ be sets with $A \leq_T B$. If 2^{-B} is left-computable and nearly computable, then A cannot be a promptly simple set.

This theorem can be strengthened in the same way: the assumption that the number is left-computable can be omitted.

Theorem

Let $A, B \subseteq \mathbb{N}$ be sets with $A \leq_T B$. If 2^{-B} is nearly computable, then A cannot be a promptly simple set.

Thanks for your attention!